Summer School

Normal Families in Complex Analysis

Julius-Maximilians-Universität Würzburg

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1.–2. Normal families in complex dynamics, I, II.

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$$f^1 = f,$$

and

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From now on, unless otherwise stated, all rational functions are assumed to have degree at least 2.

Fixed points

Definition 1

If $z_0 \in \overline{\mathbb{C}}$ and $f(z_0) = z_0$, we say that z_0 is a *fixed point* of f.

In early work on complex dynamics (ca. 1870 – 1918), classification of fixed points and the behaviour of the iterates of a function in an unspecified small neighbourhood of a fixed point were considered.

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This generally required that the iterates $f^n(z)$ have a limit as $n \to \infty$. This condition is too restrictive to be useful in more complicated situations. Global considerations became possible when a more flexible limit concept was introduced to the theory.

Introduction of normal families into complex dynamics

In 1918, Pierre Fatou and Gaston Julia independently considered the partition of $\overline{\mathbb{C}}$ into two sets as follows.

Definition 2

We define the *set of normality* or the *Fatou set* $\mathcal{F}(f)$ of f by

$$\begin{aligned} \mathcal{F}(f) &= \{z \in \overline{\mathbb{C}} : z \text{ has a neighborhood } U \text{ such that} \\ &\quad \{(f^n)|_U : n \in \mathbb{Z}^+\} \text{ is a normal family } \} \end{aligned}$$

and the *Julia set* $\mathcal{J}(f)$ of f by $\mathcal{J}(f) = \overline{\mathbb{C}} \setminus \mathcal{F}(f)$.

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The set $\mathcal{J}(f)$ is a closed subset of $\overline{\mathbb{C}}$, hence compact; $\mathcal{J}(f) \neq \emptyset$.

Summary of conclusions of complex dynamics. Complete invariance.

Theorem 3

The sets $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are **completely invariant** *under f, that is,*

$$f(\mathcal{F}(f)) \subset \mathcal{F}(f), \ f^{-1}(\mathcal{F}(f)) \subset \mathcal{F}(f),$$

$$f(\mathcal{J}(f)) \subset \mathcal{J}(f), \ f^{-1}(\mathcal{J}(f)) \subset \mathcal{J}(f).$$

In fact, for rational f, equality holds in all of the above inclusions.

Summary of conclusions of complex dynamics. Fatou and Julia sets of iterates.

Theorem 4

For each positive integer p, we have

$$\mathcal{F}(f^p) = \mathcal{F}(f), \ \mathcal{J}(f^p) = \mathcal{J}(f).$$

Summary of conclusions of complex dynamics. Components of the Fatou set.

Theorem 5

Let U be a component of $\mathcal{F}(f)$ and suppose that $n \geq 1$. Then there is a component U_n of $\mathcal{F}(f)$ such that $f^n(U) \subset U_n$. In fact, for rational f, equality holds: $f^n(U) = U_n$.

Summary of conclusions of complex dynamics. Wandering, periodic, preperiodic omponents of the Fatou set.

Let U be a component of $\mathcal{F}(f)$. Set $U_0 = U$ and for $n \geq 1$, set $U_n = f^n(U)$.

Definition 6

(i) If all U_n are distinct, we call U a **wandering domain** for f.
Otherwise, there are minimal $p \geq 0$ and $n \geq 1$ such that $U_p = U_{p+n}$.
(ii) If $p = 0$, so that $U = U_n$, we say that U is **periodic** with **period** n. If also $n = 1$, we say that U is an **invariant component** for f.
(iii) If $p \geq 1$, we say that U is **preperiodic**. Then U is not periodic, but U is an inverse image of a periodic component of f under the iterate f^p of f.

Summary of conclusions of complex dynamics. No wandering domains.

Theorem 7

Dennis Sullivan (1985)
A rational function of degree $d \geq 2$ has no wandering domains.

This answered an old question of Fatou.
The proof uses deformations of rational functions of a given degree by quasiconformal mappings, and is based on the fact that rational functions of a given degree depend on only finitely many real parameters.
Transcendental entire functions can have wandering domains (I.N. Baker).

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Since $\mathcal{F}(g) = \mathcal{F}(f^p) = \mathcal{F}(f)$, it suffices to classify the invariant components of the Fatou set of a function.

Summary of conclusions of complex dynamics. Attracting and superattacting components.

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There may be $\alpha \in V$ such that $g(\alpha) = \alpha$ and $\lim_{n\to\infty} g^n(z) = \alpha$, locally uniformly for $z \in V$. Then α is unique (in V).
Replacing $g(z)$ by $1/g(1/z)$, if necessary, we may assume that $\alpha \neq \infty$.

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Then the component V, and the fixed point α, are called **attracting** (for g) if $0 < |g'(\alpha)| < 1$, and **superattracting** if $g'(\alpha) = 0$.

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An example with $\alpha = 0$ would be $g(z) = \lambda z + z^2$ where $0 < |\lambda| < 1$, or $g(z) = z^k$ where $k \geq 2$.

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A cycle of distinct components $U = U_0, U_1, \ldots, U_{p-1}$, where $f^p(U) = U$, and U is (super)attracting for f^p, is called an **attracting/superattracting cycle** (of domains).

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If $\alpha = 0$ and $g(z) = z + az^{k+1} + O(z^{k+2})$ as $z \to 0$ ($a \neq 0$, $k \geq 1$), then there are k **petals** with 0 on their boundary where the dynamics of g can be carefully described, each petal being contained in its own component of $\mathcal{F}(g)$.

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Parabolic cycles of domains are defined analogously.

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Here W is either the unit disk, in which case the Siegel disk V contains the neutral (or indifferent) fixed point $\varphi^{-1}(0)$ of g, or W is an annulus $\{z : 1 < |z| < R\}$ centred at the origin, in which case V contains no fixed point of any iterate of g.

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An example of a function having a Herman ring (containing, incidentally, the unit circle) is given by

$$g(z) = e^{2\pi i\beta} z^2 \left(\frac{1 + \overline{a}z}{z + a} \right)$$

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We must have $\deg g \geq 3$ for it to be possible for g to have a Herman ring. Polynomials have no Herman rings, by the maximum principle.

Number of periodic cycles.

In the late 1980s, M. Shishikura developed the technique of quasiconformal surgery. This allowed him to prove the following result.

Theorem 8

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The number of critical points of f is $2d - 2$, counting multiplicities; hence f has at most $2d - 2$ distinct critical points.

Summary of conclusions of complex dynamics. Critical points.

Definition 9

If f is rational, then $z_0 \in \overline{\mathbb{C}}$ is a **critical point** of f if f is not locally homeomorphic at z_0. If $z_0 \neq \infty$, this means that $f'(z_0) = 0$, or z_0 is a multiple pole of f.
A **critical value** of f is a value $f(z_0)$, where z_0 is a critical point of f.
For a rational f, the **singularities of the inverse function** of f are the critical values of f.
The **singular set** of f is the union of all the singularities of the inverse functions of all the iterates of f.

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Sometimes the singular set is called the **postcritical set**. One often deals with the closure of the postcritical set.

Summary of conclusions of complex dynamics. Critical points.

Proposition 1

The singular set of f is given by

$$\bigcup_{j=0}^{\infty} f^j(CV(f)),$$

where $CV(f)$ is the set of all critical values of f.

Summary of conclusions of complex dynamics. Singularities determine dynamics.

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(iii) *If V is a Siegel disk or a Herman ring for any iterate of f, then every point of ∂V is a cluster point of the singular set of f.*

In fact, an attracting or parabolic invariant component of $\mathcal{F}(f)$ contains a critical point of f (so does a superattracting component, trivially). In case of a cycle, one of the components in the cycle contains a critical point of f.

Summary of conclusions of complex dynamics. Example.

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This shows that f has no attracting or parabolic periodic points and no rotation domains. Since neither 0 nor 2 is a periodic point of f, there are no superattracting periodic points for f. Hence $\mathcal{F}(f) = \emptyset$.

Summary of conclusions of complex dynamics. Branches of inverse functions.

Theorem 11

Let f be rational of degree at least 2. Let G be a domain in $\overline{\mathbb{C}}$. Suppose that $\mathcal{G}$ is a family of meromorphic functions in G such that each $g \in \mathcal{G}$ is a branch of the inverse function of some iterate of f, well defined in G. Then the family $\mathcal{G}$ is normal.

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This theorem is very helpful in obtaining the previously mentioned results indicating a connection between the singular set and the components of the Fatou set of a function.

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Proof. We will see later that there are disjoint subsets A and B of $\overline{\mathbb{C}}$, each containing at least three elements, and each consisting of a finite number of cycles $\alpha, f(\alpha), \ldots, f^{p-1}(\alpha)$ where $f^p(\alpha) = \alpha$ for some $p \geq 1$. (The value of the multiplier $(f^p)'(\alpha)$ is irrelevant.) The function f has finitely many fixed points of order 1 and 2 but infinitely many fixed points, counting all orders. Thus f has (at least) two disjoint cycles of fixed points of exact order greater than or equal to 3. These cycles then form the sets A and B.

Summary of conclusions of complex dynamics. Branches of inverse functions.

Proof, continued.

Next we note that each $g \in \mathcal{G}$ omits in $G \setminus A$ all the values in A. For if $z \in G \setminus A$ and $g(z) = a \in A$, and if g is a branch of f^{-k}, then

$$z = f^k(g(z)) = f^k(a) \in A \ ,$$

which is a contradiction. Note that any $a \in A$ belongs to a fixed point cycle that A contains completely; hence $f^n(a) \in A$ for all $n \geq 0$. Since A contains at least three points, it follows that $\mathcal{G}$ is a normal family in $G \setminus A$. Similarly, $\mathcal{G}$ is a normal family in $G \setminus B$. Since A and B are disjoint, we have

$$(G \setminus A) \cup (G \setminus B) = G \ ,$$

and so $\mathcal{G}$ is a normal family in G.

Summary of conclusions of complex dynamics. The exceptional set.

Definition 12

The **backward orbit** $O^-(z)$ of $z \in \overline{\mathbb{C}}$ under a rational function f is the set

$$O^-(z) = \{w \in \overline{\mathbb{C}} : \text{there is } n \geq 1 \text{ such that } f^n(w) = z\}.$$

Definition 13

The **exceptional set** of a rational function f is the set

$$\mathcal{E}(f) = \{z \in \overline{\mathbb{C}} : \operatorname{card} O^-(z) < \infty\}.$$

Summary of conclusions of complex dynamics. The exceptional set.

Theorem 14

If f is rational of degree ≥ 2, then $\operatorname{card}\mathcal{E}(f) \leq 2$. Furthermore, if $\mathcal{E}(f) \neq \emptyset$, then there is a Möbius transformation g such that $F = g \circ f \circ g^{-1}$ satisfies the following:
(i) If $\operatorname{card}\mathcal{E}(f) = 2$, then $F(z) = z^n$ where n is an integer with $|n| \geq 2$, and $\mathcal{E}(F) = \{0, \infty\}$;
(ii) If $\operatorname{card}\mathcal{E}(f) = 1$, then F is a polynomial and $\mathcal{E}(F) = \{\infty\}$.
Thus $\mathcal{E}(f) \subset \mathcal{F}(f)$, and $\mathcal{E}(f^p) = \mathcal{E}(f)$ for all $p \geq 1$.

Summary of conclusions of complex dynamics. Inverse images cluster to the Julia set.

Theorem 15

Suppose that $z \notin \mathcal{E}(f)$. Then the set $\mathcal{J}(f)$ is contained in the set of limit points of $O^-(z)$, that is, if $w \in \mathcal{J}(f)$ then there are sequences n_k and α_k such that $n_k \to \infty$, $f^{n_k}(\alpha_k) = z$, $\alpha_k \neq w$ and $\alpha_k \to w$ as $k \to \infty$.
In particular, if $z \in \mathcal{J}(f)$, then $\mathcal{J}(f)$ coincides with the set of limit points of $O^-(z)$, and indeed then $\mathcal{J}(f)$ is equal to the closure of $O^-(z)$.

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Theorem 15

Suppose that $z \notin \mathcal{E}(f)$. Then the set $\mathcal{J}(f)$ is contained in the set of limit points of $O^-(z)$, that is, if $w \in \mathcal{J}(f)$ then there are sequences n_k and α_k such that $n_k \to \infty$, $f^{n_k}(\alpha_k) = z$, $\alpha_k \neq w$ and $\alpha_k \to w$ as $k \to \infty$.
In particular, if $z \in \mathcal{J}(f)$, then $\mathcal{J}(f)$ coincides with the set of limit points of $O^-(z)$, and indeed then $\mathcal{J}(f)$ is equal to the closure of $O^-(z)$.

Remark. If $z \notin \mathcal{J}(f)$, then the set $O^-(z)$ might have limit points contained in $\mathcal{F}(f)$. This is the case, for example, if z belongs to a Siegel disk D but z is not the fixed point of f contained in D.

Summary of conclusions of complex dynamics. The expanding property.

Theorem 16

(The expanding property.)

If f is rational of degree ≥ 2, if $\alpha \in \mathcal{J}(f)$, and if U is a neighbourhood of α, then for each compact subset K of $\overline{\mathbb{C}} \setminus \mathcal{E}(f)$, there is a positive integer N such that for all $n \geq N$, we have

$$K \subset f^n(U).$$

In particular, if $\mathcal{E}(f) = \emptyset$, we may choose $K = \overline{\mathbb{C}}$.

Summary of conclusions of complex dynamics. Consequences.

Theorem 17

Either $\mathcal{J}(f) = \overline{\mathbb{C}}$, or $\mathcal{J}(f)$ is nowhere dense (that is, the interior of the closed set $\mathcal{J}(f)$ is empty).

Recall that a set is nowhere dense if, and only if, its closure has no interior points.

Summary of conclusions of complex dynamics. Consequences.

Theorem 17

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Recall that a set is nowhere dense if, and only if, its closure has no interior points.

Theorem 18

If $\alpha \in \mathcal{J}(f)$ and U is a neighbourhood of α, then the family

$$\left\{ f^{n_k} \mid U : k \geq 1 \right\}$$

is not normal for any subsequence n_k of distinct positive integers.

Summary of conclusions of complex dynamics. Density of repelling periodic points.

Theorem 19

The repelling fixed points of all the iterates of f lie in $\mathcal{J}(f)$, and form a dense subset of $\mathcal{J}(f)$.

Summary of conclusions of complex dynamics. Parameter spaces.

Quadratic polynomials and the Mandelbrot set.

Consider the iteration of a quadratic polynomial $F(z) = \alpha z^2 + \beta z + \gamma$ where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha \neq 0$.

Summary of conclusions of complex dynamics. Parameter spaces.

Quadratic polynomials and the Mandelbrot set.

Consider the iteration of a quadratic polynomial $F(z) = \alpha z^2 + \beta z + \gamma$ where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha \neq 0$.

There are complex numbers A, B with $A \neq 0$ such that with $g(z) = Az + B$, we have

$$(g \circ F \circ g^{-1})(z) = z^2 + c \equiv f_c(z)$$

for some $c \in \mathbb{C}$.

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for some $c \in \mathbb{C}$.

Since the behaviour of F and $g \circ F \circ g^{-1}$ is substantially the same under iteration (we have $g \circ F^n \circ g^{-1} = (g \circ F \circ g^{-1})^n$), it suffices to consider functions of the form f_c.

Summary of conclusions of complex dynamics. Parameter spaces.

The origin is the only finite critical point of $f_c = z^2 + c$.

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The forward orbit of 0 under f_c is given by the polynomials $p_n(c)$, where

$$p_1(c) = c,\ \ p_{n+1}(c) = p_n(c)^2 + c.$$

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Thus the $p_n(c)$ are

$$c, \ \ c^2 + c, \ \ (c^2 + c)^2 + c, \ldots$$

Summary of conclusions of complex dynamics. Parameter spaces.

One can prove that the Julia set $\mathcal{J}(f_c)$ is connected if, and only if, the forward orbit of the critical point 0 is bounded.

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$$\mathcal{M} = \{c \in \mathbb{C} : |p_n(c)| \leq 2 \ \forall n \geq 1\}.$$

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A. Douady and J.H. Hubbard proved in 1984 that $\mathcal{M}$ is connected.

Summary of conclusions of complex dynamics. Parameter spaces.

The maximal open set V where the family $\{p_n : n \geq 1\}$ is normal has an unbounded component U, where, in fact, $p_n(c) \to \infty$.

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A **hyperbolic component** of the interior of $\mathcal{M}$ is a bounded component W of V such that for all $c \in W$, the function f_c has a periodic cycle of some period p (attracting, or, for exactly one $c \in W$, superattracting).

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Open conjecture. *Every component of the interior of $\mathcal{M}$ is hyperbolic.*

Summary of conclusions of complex dynamics. Parameter spaces.

Other parameter spaces may be considered, for other families of polynomials or rational functions, or for transcendental entire functions such as $e^{\lambda z}$ or λe^z.

A problem going back to Euler asked about the convergence of a sequence that may be written as $f^n(0)$, where $f(z) = e^{az}$.
I.N. Baker and P.J. Rippon (1983) proved the following result.

Theorem 20

Suppose that $a \in \mathbb{C}$, $b = e^a$, $w_1 = b$, and $w_{n+1} = b^{w_n} \equiv e^{aw_n}$ for $n \geq 1$. The sequence w_n has a limit if $a = te^{-t}$ where $t \in \mathbb{C}$, $|t| < 1$, or t is a root of unity.
For almost all t with $|t| = 1$, the sequence w_n diverges.

Summary of conclusions of complex dynamics. Example.

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For almost all t with $|t| = 1$, the sequence w_n diverges.

On the real axis, if $-1 < t < 1$, then $a = te^{-t}$ varies from $-e$ to $1/e$, and $b = e^a$ varies from e^{-e} to $e^{1/e}$. Euler's problem was to prove that w_n converges for these b, that is, for $e^{-e} < b < e^{1/e}$.

If f is entire, then $c \in \overline{\mathbb{C}}$ is an **asymptotic value** of f if there is a path γ going to infinity such that $f(z) \to c$ as $z \to \infty$ along γ.

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An essential point is that for the function $f(z) = e^{az}$, the singularities of f^{-1} are the critical values of f (none, since f' never vanishes) and the asymptotic values, here 0 and ∞. Thus the only finite singularity of f^{-1} is 0, which is where we may start the iteration.

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Here the extra structure provided by the Fatou–Julia theory of dynamics can be used to solve a problem that originally (Euler) only referred to real numbers.

Summary of conclusions of complex dynamics, continued

This is the end of our summary of conclusions of complex dynamics.

We now proceed to consider complex dynamics in greater detail, also giving a number of proofs, and keeping in mind the definitions already made.

Classification of fixed points

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If $z_0 \neq \infty$, we say that z_0 is
(i) superattracting if $f'(z_0) = 0$,
(ii) attracting if $0 < |f'(z_0)| < 1$,
(iii) parabolic if $f'(z_0) = 1$.

Classification of fixed points

If $f(z_0) = z_0 \neq \infty$, we say that z_0 is
(iv) neutral or *indifferent* if $|f'(z_0)| = 1$,
(v) rationally neutral or *indifferent* if $f'(z_0)$ is a root of unity,
(vi) irrationally neutral or *indifferent* if $|f'(z_0)| = 1$, but $f'(z_0)$ is not a root of unity,
(vii) repelling if $|f'(z_0)| > 1$,
(viii) weakly repelling if either $f'(z_0) = 1$ or $|f'(z_0)| > 1$.

Classification of fixed points

If $z_0 = \infty$, we determine the type of the fixed point by considering $1/f(1/z)$ at the origin.
The type of fixed point (as determined by the actual behaviour of the f^n) is preserved by any conformal conjugation, thus allowing us to consider only the case $z_0 = 0$.

Classification of fixed points

For every non-neutral fixed point z_0 of f, we can find a conformal map ϕ with $\phi(z_0) = 0$ and $\phi'(z_0) = 1$, defined in a neighborhood of z_0 such that

$$(\phi \circ f \circ \phi^{-1})(z) = f'(z_0)z$$

if $0 < |f'(z_0)| < 1$ or $|f'(z_0)| > 1$, and

$$(\phi \circ f \circ \phi^{-1})(z) = z^k$$

if

$$f(z_0) = z_0 + a(z - z_0)^k + \cdots, a \neq 0, k \geq 2.$$

If $|f'(z_0)| < 1$, it is easy to show that there is $\rho > 0$ such that $f(\overline{B}(z_0, \rho)) \subset B(z_0, \rho)$ and hence $f^n(z) \to z_0$ as $n \to \infty$, uniformly for $z \in \overline{B}(z_0, \rho)$.

Invariance properties of sets, defined.

Definition 21

We say that a set $E \subset \overline{\mathbb{C}}$ is
(i) (forward) **invariant** under f if $f(E) \subset E$,
(ii) **backward invariant** under f if $f^{-1}(E) \subset E$,
(iii) **completely invariant** if both (i) and (ii) hold.

A property of normal families.

Theorem 22

Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \overline{\mathbb{C}}$. Suppose that $0 < a < b$. Then $\mathcal{F}$ is a normal family if, and only if, each $z_0 \in D$ has a neighbourhood U such that each $f \in \mathcal{F}$ satisfies $|f(z)| > a$ for all $z \in U$, or $|f(z)| < b$ for all $z \in U$.

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Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \overline{\mathbb{C}}$. Suppose that $0 < a < b$. Then $\mathcal{F}$ is a normal family if, and only if, each $z_0 \in D$ has a neighbourhood U such that each $f \in \mathcal{F}$ satisfies $|f(z)| > a$ for all $z \in U$, or $|f(z)| < b$ for all $z \in U$.

Proof. Let $\mathcal{F}$ be normal. Suppose $z_0 \in D$. If z_0 has no neighbourhood U as required, then there are $f_n \in \mathcal{F}$ and points $z_n, w_n \to z_0$ such that $|f_n(z_n)| \leq a$ and $|f_n(w_n)| \geq b$. By passing to a subsequence without changing notation, we may assume that $f_n \to f$ locally uniformly in D. Then $|f(z_0)| = \lim_{n\to\infty} |f_n(z_n)| \leq a$, and $|f(z_0)| = \lim_{n\to\infty} |f_n(w_n)| \geq b$. This is a contradiction since $a < b$. Hence, if $\mathcal{F}$ is normal, each $z_0 \in D$ has a neighbourhood U with the required property.

A property of normal families, continued.

Proof, continued. Suppose that each $z_0 \in D$ has a neighbourhood U with the required property. Let $f_n \in \mathcal{F}$. By passing to a subsequence without changing notation, we may assume that we have $|f_n| > a$ in U for all n, or that we have $|f_n| < b$ in U for all n.
In the former case, the $1/f_n$ are uniformly bounded in U, so that the sequence f_n has a locally uniformly convergent subsequence.
In the latter case, the functions f_n are uniformly bounded in U, so that the sequence f_n has a locally uniformly convergent subsequence.
Hence $\mathcal{F}$ is normal at z_0. Since this is true for each $z_0 \in D$, it follows that $\mathcal{F}$ is normal in D.
This completes the proof of this theorem.

Complete invariance of the Fatou and Julia sets.

Theorem 23

Both $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant under f.

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Both $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant under f.

Proof. It suffices to prove the inclusions for $\mathcal{F}(f)$. Suppose $\alpha \in \mathcal{F}(f)$, and let U be a neighbourhood of α with $\overline{U} \subset \mathcal{F}(f)$ such that each f^n satisfies $|f^n(z)| < 2$ or $|f^n(z)| > 1$ throughout U.

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To prove that $f(\mathcal{F}(f)) \subset \mathcal{F}(f)$, we need to show that $f(\alpha) \in \mathcal{F}(f)$. Since f is an open mapping, it follows that $f(U)$ is a neighbourhood of $f(\alpha)$. We claim that the family of functions $\{f^n \mid f(U) : n \geq 1\}$ is normal. Each point $z \in f(U)$ has a neighbourhood, namely $f(U)$ itself, such that for every $n \geq 1$, we have $|f^n(z)| < 2$ or $|f^n(z)| > 1$ throughout $f(U)$. Hence the family $\{f^n \mid f(U) : n \geq 1\}$ is normal. Thus $f(\alpha) \in \mathcal{F}(f)$ and so $f(\mathcal{F}(f)) \subset \mathcal{F}(f)$.

Proof, continued.

If $f(\beta) = \alpha$ and if U is as above, then β has a neighbourhood V such that $f(V) \subset U$. Clearly the family $\{f^n \mid V : n \geq 1\}$ is normal. Hence $\beta \in \mathcal{F}(f)$, and so $f^{-1}(\mathcal{F}(f)) \subset \mathcal{F}(f)$. This completes the proof.

An inclusion relation for Fatou and Julia sets.

Proposition 2

We have $\mathcal{F}(f) \subset \mathcal{F}(f^p)$*, for every* $p \geq 1$*, and therefore* $\mathcal{J}(f) \supset \mathcal{J}(f^p)$*.*

An inclusion relation for Fatou and Julia sets.

Proposition 2

We have $\mathcal{F}(f) \subset \mathcal{F}(f^p)$, for every $p \geq 1$, and therefore $\mathcal{J}(f) \supset \mathcal{J}(f^p)$.

Proof. If $z \in \mathcal{F}(f)$, then z has a neighbourhood U in which the family $\{f^n : n \geq 1\}$ is normal, so that the smaller family $\{f^{np} : n \geq 1\}$ is also normal in U. Hence $z \in \mathcal{F}(f^p)$, and so $\mathcal{F}(f) \subset \mathcal{F}(f^p)$.

Julia sets are perfect.

Theorem 24

The Julia set $\mathcal{J}(f)$ of a rational function f with $\deg(f) \geq 2$ is perfect and hence uncountable.
There are infinitely many distinct weakly repelling fixed points of all the iterates f^n of f.

Julia sets are perfect.

Theorem 24

The Julia set $\mathcal{J}(f)$ of a rational function f with $\deg(f) \geq 2$ is perfect and hence uncountable.
There are infinitely many distinct weakly repelling fixed points of all the iterates f^n of f.

Proof. First, we show $\mathcal{J}(f) \neq \emptyset$. We claim that any weakly repelling periodic point is in $\mathcal{J}(f)$. If $g = f^p$ for some $p \geq 1$ and if $g(\alpha) = \alpha$ and α is weakly repelling, then g has the local power series expansion

$$g(z) = \alpha + \lambda(z - \alpha) + a(z - \alpha)^k + O((z - \alpha)^{k+1}),$$

where $\lambda = 1$ or $|\lambda| > 1$, and $a \neq 0$ and $k \geq 2$, since $\deg f \geq 2$.

Julia sets are perfect.

Proof, continued.
If $|\lambda| > 1$ then

$$g^n(z) = \alpha + \lambda^n(z - \alpha) + O((z - \alpha)^k),$$

so that the functions g^n do not form a normal family in any neighbourhood of α.

Julia sets are perfect.

Proof, continued.
If $|\lambda| > 1$ then

$$g^n(z) = \alpha + \lambda^n(z-\alpha) + O((z-\alpha)^k),$$

so that the functions g^n do not form a normal family in any neighbourhood of α.

If $\lambda = 1$ then

$$g^n(z) = \alpha + (z-\alpha) + na(z-\alpha)^k + O((z-\alpha)^{k+1}),$$

so that the functions g^n do not form a normal family in any neighbourhood of α. Therefore $\alpha \in \mathcal{J}(g) = \mathcal{J}(f^p) \subset \mathcal{J}(f)$.

Proof, continued.

We may assume that $f^n(\infty) \neq \infty$ for all $n \geq 1$, for we can replace f by $M \circ f \circ M^{-1}$ for some Möbius transformation M, if necessary, to achieve this.

Proof, continued.

Now, $\deg(g) = d^p$, where $\deg f = d$, so g has $d^p + 1 \geq 3$ fixed points, with due count of multiplicity. We claim that at least one fixed point of g is weakly repelling, therefore in $\mathcal{J}(f)$. Suppose that this is false. Then if $R > 0$ is large enough, it follows from the residue theorem that

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z - f^p(z)} = \frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z - g(z)}$$

$$= \sum_{g(\alpha)=\alpha}\frac{1}{1 - g'(\alpha)}.$$

Proof, continued.

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z-f^p(z)}=\sum_{g(\alpha)=\alpha}\frac{1}{1-g'(\alpha)}.$$

The integral on the left tends to 1 as $R\to\infty$. We assumed that no fixed point of g is weakly repelling, so we must have $\lambda=g'(\alpha)=a+ib\neq 1$ and $|\lambda|=a^2+b^2\leq 1$, for every fixed point α of g. Also each fixed point α of g is a simple zero of $g(z)-z$ since $g'(\alpha)\neq 1$. Taking the real part of the sum on the right hand side, we get for each term the quantity

$$\mathrm{Re}\left[\frac{1}{1-\lambda}\right]=\frac{1-a}{(1-a)^2+b^2}\geq\frac{1}{2}.$$

Here $1-a\neq 0$, for otherwise we get $\lambda=1$.

Proof, continued.

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z-f^p(z)}=\sum_{g(\alpha)=\alpha}\frac{1}{1-g'(\alpha)}.$$

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$$\mathrm{Re}\left[\frac{1}{1-\lambda}\right]=\frac{1-a}{(1-a)^2+b^2}\geq\frac{1}{2}.$$

Here $1-a\neq 0$, for otherwise we get $\lambda=1$.

Hence the real part of the sum satisfies

$$\mathrm{Re}\sum_{g(\alpha)=\alpha}\frac{1}{1-g'(\alpha)}\geq\sum_{g(\alpha)=\alpha}\frac{1}{2}\geq\frac{3}{2}.$$

This contradicts the previous result that the integral on the left hand side tends to 1 as $R\to\infty$.

Proof, continued.

Therefore there is at least one point $\alpha \in \mathcal{J}(f)$, where $f^p(\alpha) = \alpha$ for some $p \geq 1$, and furthermore $(f^p)'(\alpha) = 1$ or $|(f^p)'(\alpha)| > 1$.

The **orbit** of α satisfies $\mathcal{O}(\alpha) \subset \mathcal{J}(f)$, where $\mathcal{O}(\alpha)$ is defined by

$$\mathcal{O}(\alpha) = \{w \in \mathbb{C} : f^n(w) = \alpha,$$

$$\text{or } f^n(\alpha) = w, \text{ for some integer } n \geq 1\}.$$

Proof, continued.

Therefore there is at least one point $\alpha \in \mathcal{J}(f)$, where $f^p(\alpha) = \alpha$ for some $p \geq 1$, and furthermore $(f^p)'(\alpha) = 1$ or $|(f^p)'(\alpha)| > 1$.

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$$\mathcal{O}(\alpha) = \{w \in \mathbb{C} : f^n(w) = \alpha,$$

$$\text{or } f^n(\alpha) = w, \text{ for some integer } n \geq 1\}.$$

If $\mathrm{card}(\mathcal{O}(\alpha)) \leq 2$, one can check that for some Möbius map M, the function $M \circ f \circ M^{-1}$ is a polynomial, or a map $(M \circ f \circ M^{-1})(z) = cz^n$, where $c \neq 0$, $|n| \geq 2$. One can check separately that $\mathcal{J}(cz^n)$ is a circle, which, indeed, is a perfect set. If $M \circ f \circ M^{-1}$ is a polynomial, the fixed point $\alpha = M^{-1}(\infty)$ is superattracting, therefore in $\mathcal{F}(f)$. This is a contradiction. Hence, we may assume that $\mathrm{card}(\mathcal{J}(f)) \geq \mathrm{card}(\mathcal{O}(\alpha)) \geq 3$.

Proof, continued.

Now, to get the contradiction suppose that $\beta \in \mathcal{J}(f)$ is an isolated point. Let U be a neighborhood of β with $U \setminus \{\beta\} \subset \mathcal{F}(f)$. Then each f^n omits each point of $\mathcal{J}(f)$ in $U \setminus \{\beta\}$. There are at least three points in $\mathcal{J}(f)$. Hence each f^n omits at least 3 points in $U \setminus \{\beta\}$. Thus the extended Montel's theorem applies so that $\{f^n : n \geq 1\}$ is a normal family in U. In other words, $\beta \in \mathcal{F}(f)$. This contradicts the assumption that $\beta \in \mathcal{J}(f)$ and shows that $\mathcal{J}(f)$ is perfect.

The above argument can be adapted to show that there are infinitely many distinct weakly repelling fixed points of all the iterates of f. We omit the details.

An extension of Montel's theorem

In the previous proof, we used the following result.

Theorem 25

Let the family $\mathcal{F}'$ of meromorphic functions in the punctured disk $D' = \{z : 0 < |z| < \delta\}$ have the property that each $f \in \mathcal{F}'$ omits in D' certain three fixed values in $\overline{\mathbb{C}}$. Then each $f \in \mathcal{F}'$ can be extended to be meromorphic in $D = \{z : |z| < \delta\}$, and the family $\mathcal{F}$ of these extended functions is normal in D.

An extension of Montel's theorem, proof

Proof. The Big Picard Theorem implies that each $f \in \mathcal{F}'$ can be extended to be meromorphic in D. We may assume that the omitted values are $0, 1, \infty$. Also we know that $\mathcal{F}'$ is normal in D'. Pick a sequence F_n from $\mathcal{F}$ and assume that F_n is the extension of $f_n \in \mathcal{F}'$. We may pass to a subsequence several times without changing notation, to make certain assumptions.

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First, we assume that $f_n \to f$ where f is meromorphic in D'. Either f is constant (possibly ∞) or f omits $0, 1, \infty$ in D' and is hence analytic.

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First, we assume that $f_n \to f$ where f is meromorphic in D'. Either f is constant (possibly ∞) or f omits $0, 1, \infty$ in D' and is hence analytic.

Each F_n takes a finite value or the value ∞ at $z = 0$. We may assume that $F_n(0)$ is finite for all n, or that $F_n(0) = \infty$ for all n.

An extension of Montel's theorem, proof

Case 1. f is analytic and each $F_n(0)$ is finite.
On the circle $S(0, \delta/2)$, the maximum moduli of F_n are uniformly bounded for large n (since F_n is uniformly close to f there). By the maximum modulus principle, the F_n are uniformly bounded in the disk $B(0, \delta/2)$ so that they form a normal family there.

An extension of Montel's theorem, proof

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Case 2. f is analytic and each $F_n(0) = \infty$.
If $f \not\equiv 0$ then f omits 0 in D' so that $1/f$ is analytic in D'. Now we proceed as in Case 1, considering $1/f$ and $1/F_n$ instead of f and F_n.

An extension of Montel's theorem, proof

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If $f \not\equiv 0$ then f omits 0 in D' so that $1/f$ is analytic in D'. Now we proceed as in Case 1, considering $1/f$ and $1/F_n$ instead of f and F_n.

If $f \equiv 0$, consider $1/(1-f)$ and $1/(1-F_n)$ instead of f and F_n and proceed as in Case 1.

An extension of Montel's theorem, proof

Case 3. $f \equiv \infty$, and each $F_n(0)$ is non-zero (possibly ∞).
Consider $1/f$ and $1/F_n$ instead of f and F_n and proceed as in Case 1.

An extension of Montel's theorem, proof

Case 3. $f \equiv \infty$, and each $F_n(0)$ is non-zero (possibly ∞).
Consider $1/f$ and $1/F_n$ instead of f and F_n and proceed as in Case 1.

Case 4. $f \equiv \infty$ and each $F_n(0) = 0$.
Consider $1/(1-f)$ and $1/(1-F_n)$ instead of f and F_n and proceed as in Case 1.

A further extension of Montel's theorem

Theorem 26

Let D be a domain in $\overline{\mathbb{C}}$ and let $\mathcal{F}$ be a family of meromorphic functions in $D \setminus \{\alpha\}$, where $\alpha \in D$. Suppose that there is a positive number δ depending on $\mathcal{F}$ and D only and suppose that for each $f \in \mathcal{F}$ there are distinct points $a(f)$, $b(f)$ and $c(f)$ in $\overline{\mathbb{C}}$ which f omits in $D \setminus \{\alpha\}$, such that

$$\delta \leq \min\{q(a(f), b(f)), q(b(f), c(f)), q(c(f), a(f))\}.$$

Then each $f \in \mathcal{F}$ can be extended to a meromorphic function $\Phi(f)$ in D, and the family $\{\Phi(f) : f \in \mathcal{F}\}$ is a normal family in D.

Here

$$q(z, w) = \frac{|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$

is the chordal distance of $z, w \in \overline{\mathbb{C}}$.

Julia's definition of what became known as the Julia set

Theorem 27

If f is rational, then $\mathcal{J}(f)$ is the smallest subset of $\overline{\mathbb{C}}$ that is closed, backward invariant under f, and contains at least 3 points.

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Proof. Let E be a closed subset of $\overline{\mathbb{C}}$ that is closed, backward invariant under f, and contains at least 3 points. Then $U = \overline{\mathbb{C}} \setminus E$ is open, and U is forward invariant under f. Hence, in U, all the iterates f^n of f omit all points of E, and hence omit at least 3 points. Thus $\{f^n|U : n \geq 1\}$ is normal, so that $U \subset \mathcal{F}(f)$. Thus E contains $\mathcal{J}(f)$.

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We know that $\mathcal{J}(f)$ is closed, backward invariant under f, and contains at least 3 points. Hence $\mathcal{J}(f)$ is the smallest set with these properties.

Components of the Fatou set map onto components.

Proposition 3

If U is a component of $\mathcal{F}(f)$ then $f(U)$ is equal to a component of $\mathcal{F}(f)$.

Proof. In any case $f(U) \subset V$ for some component V of $\mathcal{F}(f)$, because f is continuous on $\mathbb{C}$. Suppose that $V \setminus f(U) \neq \emptyset$. Then we can find a point $a \in V \cap \partial f(U)$. Let a_n be a sequence in $f(U)$ that converges to a. Then there is a sequence b_n in U, such that $f(b_n) = a_n$ for all n. By passing to a subsequence without changing notation, we may assume that $b_n \to b$, for some $b \in \overline{U}$, and

$$f(b) = f(\lim_{n\to\infty} b_n) = \lim_{n\to\infty} f(b_n) = a.$$

Proof, continued.

We claim that $b \notin U$. If $b \in U$, then $a \in f(U)$. The function f is non-constant and meromorphic so that $f(U)$ is open, yet we found $a \in f(U) \cap \partial f(U)$, which is impossible. Therefore, we must have $b \in \partial U \subset \mathcal{J}(f)$ so that $a = f(b) \in \mathcal{J}(f)$. Since a was arbitrary, we get $\partial f(U) \cap V = \emptyset$, while $f(U) \subset V$. This, coupled with the fact that both $f(U)$ and V are open, implies that $f(U) = V$.

Inverse images cluster to the Julia set

Theorem 28

Suppose that $z \notin \mathcal{E}(f)$. Then the set $\mathcal{J}(f)$ is contained in the set of limit points of $O^-(z)$, that is, if $w \in \mathcal{J}(f)$ then there are sequences n_k and α_k such that $n_k \to \infty$, $f^{n_k}(\alpha_k) = z$, $\alpha_k \neq w$ and $\alpha_k \to w$ as $k \to \infty$.
In particular, if $z \in \mathcal{J}(f)$, then $\mathcal{J}(f)$ coincides with the set of limit points of $O^-(z)$, and indeed then $\mathcal{J}(f)$ is equal to the closure of $O^-(z)$.

Inverse images cluster to the Julia set

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In particular, if $z \in \mathcal{J}(f)$, then $\mathcal{J}(f)$ coincides with the set of limit points of $O^-(z)$, and indeed then $\mathcal{J}(f)$ is equal to the closure of $O^-(z)$.

Proof. Suppose that $z \in \overline{\mathbb{C}} \setminus \mathcal{E}(f)$ so that $O^-(z)$ is an infinite set, and that $w \in \mathcal{J}(f)$. Let U be a neighbourhood of w. We need to show that $U \setminus \{w\}$ intersects $O^-(z)$. If not, then each f^n omits all the values in $O^-(z)$ in $U \setminus \{w\}$ so that the family $\{f^n|(U \setminus \{w\}) : n \geq 1\}$ is normal and so $U \setminus \{w\} \subset \mathcal{F}(f)$. Since $\mathcal{J}(f)$ is a perfect set, this is impossible.
If $z \in \mathcal{J}(f)$ then $O^-(z) \subset \mathcal{J}(f)$ since $\mathcal{J}(f)$ is completely invariant under f. Since $\mathcal{J}(f)$ is closed, it follows that all the limit points of $O^-(z)$ belong to $\mathcal{J}(f)$, and further that $\overline{O^-(z)} \subset \mathcal{J}(f)$. Thus $\mathcal{J}(f)$ coincides with the set of such limit points, as well as with the closure of $O^-(z)$.

The expanding property.

Theorem 29

(The expanding property.)
If f is rational of degree ≥ 2, if $\alpha \in \mathcal{J}(f)$, and if U is a neighbourhood of α, then for each compact subset K of $\overline{\mathbb{C}} \setminus \mathcal{E}(f)$, there is a positive integer N such that for all $n \geq N$, we have

$$K \subset f^n(U).$$

In particular, if $\mathcal{E}(f) = \emptyset$, we may choose $K = \overline{\mathbb{C}}$.

The expanding property, proof.

We may assume that $\alpha \neq \infty$ and therefore that $U \subset \mathbb{C}$.

The expanding property, proof.

We may assume that $\alpha \neq \infty$ and therefore that $U \subset \mathbb{C}$.
If $z \in K$ then by a previous Theorem, there is $\beta \in U$ such that $f^n(\beta) = z$ for some $n \geq 1$. Thus

$$z \in \bigcup_{n=1}^{\infty} f^n(U)$$

and since z was arbitrary, we have

$$K \subset \bigcup_{n=1}^{\infty} f^n(U) \ .$$

Since K is compact, this gives

$$K \subset \bigcup_{n=1}^{N} f^n(U)$$

for some $N \geq 1$. However, the result that $K \subset f^n(U)$ for all large n, lies deeper.

The expanding property, proof, continued.

Since $\mathcal{J}(f)$ is perfect, we can choose points $\alpha_i \in \mathcal{J}(f) \cap U$ for $1 \leq i \leq 3$ and a positive number ρ so that the disks

$$\Delta_i = \{z : |z - \alpha_i| < 2\rho\} \quad \text{for} \quad 1 \leq i \leq 3$$

have disjoint closures $\overline{\Delta_i}$ contained in $U \setminus \mathcal{E}(f)$. We define

$$D_i = \{z : |z - \alpha_i| < \rho\} \quad \text{for} \quad 1 \leq i \leq 3 \ .$$

The expanding property, proof, continued.

Since $\mathcal{J}(f)$ is perfect, we can choose points $\alpha_i \in \mathcal{J}(f) \cap U$ for $1 \leq i \leq 3$ and a positive number ρ so that the disks

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have disjoint closures $\overline{\Delta_i}$ contained in $U \setminus \mathcal{E}(f)$. We define

$$D_i = \{z : |z - \alpha_i| < \rho\} \quad \text{for} \quad 1 \leq i \leq 3 \, .$$

Suppose that for each large $n \geq 1$, we have

$$\Delta_i \setminus f^n(D_1) \neq \emptyset \quad \text{for} \quad 1 \leq i \leq 3 \, ,$$

that is, $f^n \mid D_1$ omits some value in each Δ_i. Thus the family $\{f^n \mid D_1 : n \geq 1\}$ is normal so that $\alpha_1 \in \mathcal{F}(f)$, which is impossible. It follows that for each $n \geq 1$ in an infinite sequence, there is i with $1 \leq i \leq 3$ such that

$$(*) \qquad \Delta_i \subset f^n(D_1) \, .$$

Let us take some fixed values n and i in (*).

The expanding property, proof, continued.

In the same way we obtain integers m, p, j and k with $m \geq 1$, $p \geq 1$, $1 \leq j \leq 3$ and $1 \leq k \leq 3$ such that

$$(**) \qquad \Delta_j \subset f^m(D_2) \ \text{ and } \ \Delta_k \subset f^p(D_3) \ .$$

We claim that there are integers $q \geq 1$ and $\ell \in \{1, 2, 3\}$ such that

$$(***) \qquad \Delta_\ell \subset f^q(D_\ell) \ .$$

The expanding property, proof, continued.

In the same way we obtain integers m, p, j and k with $m \geq 1$, $p \geq 1$, $1 \leq j \leq 3$ and $1 \leq k \leq 3$ such that

$$(**) \qquad \Delta_j \subset f^m(D_2) \text{ and } \Delta_k \subset f^p(D_3) .$$

We claim that there are integers $q \geq 1$ and $\ell \in \{1, 2, 3\}$ such that

$$(***) \qquad \Delta_\ell \subset f^q(D_\ell) .$$

A simple reasoning that considers all possibilities that can arise, proves that (***) follows from (*) and (**). We shall give the full proof here for the sake of completeness.

The expanding property, proof, continued.

If $i = 1$ or $j = 2$ or $k = 3$, then (***) obviously holds. Otherwise, set $i_1 = i$, $i_2 = j$, $i_3 = k$, $n_1 = n$, $n_2 = m$ and $n_3 = p$. There may be distinct integers $M, N \in \{1, 2, 3\}$ such that

$$i_M = N \quad \text{and} \quad i_N = M \ .$$

Then

$$\Delta_{i_M} = \Delta_N \subset f^{n_M}(D_M),$$
$$\Delta_{i_N} = \Delta_M \subset f^{n_N}(D_N) \ .$$

Since $D_N \subset \Delta_N$, we get

$$\Delta_M \subset f^{n_N}(D_N) \subset f^{n_N}(\Delta_N) \subset f^{n_N + n_M}(D_M) \ ,$$

which gives (***).

The expanding property, proof, continued.

If there are no M and N as above, then the integers i, j and k are distinct and so (i, j, k) is equal to $(2, 3, 1)$ or $(3, 1, 2)$. In the former case, we have

$$\Delta_2 \subset f^n(D_1), \Delta_3 \subset f^m(D_2), \Delta_1 \subset f^p(D_3)$$

so that, for example,

$$\begin{aligned} \Delta_1 \subset f^p(D_3) \subset f^p(\Delta_3) \subset f^{p+m}(D_2) \\ \subset f^{p+m}(\Delta_2) \subset f^{p+m+n}(D_1) \ . \end{aligned}$$

In the latter case, we have

$$\Delta_3 \subset f^n(D_1), \Delta_1 \subset f^m(D_2), \Delta_2 \subset f^p(D_3)$$

so that, for example,

$$\begin{aligned} \Delta_1 \subset f^m(D_2) \subset f^m(\Delta_2) \subset f^{m+p}(D_3) \\ \subset f^{m+p}(\Delta_3) \subset f^{m+p+n}(D_1) \ . \end{aligned}$$

Thus it is always possible to satisfy (***).

Let us write Δ for Δ_ℓ and D for D_ℓ in (***). Then we have

$$(+) \qquad \Delta \subset f^q(D) \subset f^q(\Delta)$$

since $D \subset \Delta$.

The expanding property, proof, continued.

Let us write Δ for Δ_ℓ and D for D_ℓ in (***). Then we have

$$(+) \qquad \Delta \subset f^q(D) \subset f^q(\Delta)$$

since $D \subset \Delta$.

Next we claim that Δ intersects $\mathcal{J}(f^q)$. If not, then Δ is contained in a component G of $\mathcal{F}(f^q)$. Since $\mathcal{F}(f^q)$ is completely invariant under f^q, it follows that $f^q(G)$ is contained in a component G' of $\mathcal{F}(f^q)$. By $(+)$ we get $G \cap G' \neq \emptyset$ and so $G' = G$. Note that $\overline{\mathbb{C}} \setminus G$ contains at least three points since $\mathcal{J}(f^q)$ does.

The expanding property, proof, continued.

If a subsequence f^{qn_k} has a constant limit in G then

$$\operatorname{diam} f^{qn_k}(\,\overline{D}\,) \to 0 \quad \text{as} \quad k \to \infty$$

since $\overline{D} \subset \Delta \subset G$. This is impossible since

$$(++) \qquad \Delta \subset f^{qn}(D) \subset f^{qn}(\,\overline{D}\,)$$

for all $n \geq 1$. Thus all limit functions of convergent sequences f^{qn_k} are nonconstant, and so there is a sequence m_k tending to infinity such that $f^{qm_k} \to Id$ on G as $k \to \infty$, the convergence being uniform on $\overline{D}$. But this contradicts $(++)$ since Δ is twice the size of D. It follows that $\Delta \cap \mathcal{J}(f^q) \neq \emptyset$, as asserted.

The expanding property, proof, continued.

Now we apply a previous theorem, on points of $\mathcal{J}(f)$ being cluster points of backward orbits, to f^q instead of f and recall that $\mathcal{E}(f^q) = \mathcal{E}(f)$. Since Δ is a neighbourhood of some point in $\mathcal{J}(f^q)$ by what was shown above, we may proceed as in the remark made at the beginning of the proof and deduce that

$$K \subset \bigcup_{n=1}^{p} f^{nq}(\Delta)$$

for some $p \geq 1$. By (***), we have

$$\Delta \subset f^q(\Delta) \subset f^{2q}(\Delta) \subset \cdots$$

and so

$$(+-) \qquad K \subset f^{nq}(\Delta) \quad \text{for all} \quad n \geq p\ .$$

The expanding property, proof, continued.

Since $\Delta \cap \mathcal{E}(f) = \emptyset$ by construction, we have $f^j(\Delta) \cap \mathcal{E}(f) = \emptyset$ for all $j \geq 0$. Thus we may apply the above reasoning to the compact subset $\overline{f^j(\Delta)}$ of $\overline{\mathbb{C}} \setminus \mathcal{E}(f)$ instead of K for each j with $0 \leq j \leq q$. We deduce that there is a positive integer s such that

$$(--) \qquad \bigcup_{j=0}^{q} f^j(\Delta) \subset \bigcup_{j=0}^{q} \overline{f^j(\Delta)} \subset f^{vq}(\Delta)$$

for all $v \geq s$.

The expanding property, proof, continued.

Since $\Delta \cap \mathcal{E}(f) = \emptyset$ by construction, we have $f^j(\Delta) \cap \mathcal{E}(f) = \emptyset$ for all $j \geq 0$. Thus we may apply the above reasoning to the compact subset $\overline{f^j(\Delta)}$ of $\overline{\mathbb{C}} \setminus \mathcal{E}(f)$ instead of K for each j with $0 \leq j \leq q$. We deduce that there is a positive integer s such that

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for all $v \geq s$.

Define $N = 1 + q(p+s)$. Then any integer $n \geq N$ can be written as

$$n = tq + sq + u \quad \text{where} \quad t \geq p \quad \text{and} \quad 1 \leq u \leq q \; .$$

Consequently, since $t + 1 \geq p$, we have

$$K \subset f^{(t+1)q}(\Delta) = f^{tq}\Big\{ f^u \big[f^{q-u}(\Delta) \big] \Big\}$$

$$\subset f^{tq}\Big\{ f^u \big[f^{sq}(\Delta) \big] \Big\} = f^n(\Delta) \subset f^n(U) \; .$$

by (–) with $v = s$. This proves the Expanding Property.

Alternative ideas for the proof.

Suppose that we were to choose 5 rather than 3 domains Δ_i and D_i. Since the functions f^n do not form a normal family in D_i, it follows from results of Ahlfors related to the Ahlfors five-islands theorem, that some subdomain of D_i is mapped by some f^n conformally onto some Δ_j. This works for each i, $1 \leq i \leq 5$, with $n \geq 1$ and j with $1 \leq j \leq 5$ depending on i.

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Going around chains as in the previous proof, we then find some i and some q such that f^q maps some subdomain of D_i conformally onto Δ_i.

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Going around chains as in the previous proof, we then find some i and some q such that f^q maps some subdomain of D_i conformally onto Δ_i.

After that, we could again proceed as in the preceding proof.

In fact, this type of an argument can be used to prove that repelling periodic points are dense in $\mathcal{J}(f)$.

The Euler characteristic. Let D_i, for $1 \leq i \leq q$, be Jordan domains with disjoint closures in $\overline{\mathbb{C}}$. Set $U = \overline{\mathbb{C}} \setminus \cup_{i=1}^{q} \overline{D_i}$. The the Euler characteristic $\chi(U)$ of U is $2 - q$.

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Theorem 30

(The Riemann–Hurwitz formula)
Let X and Y be surfaces with finite Euler characteristics, and let $f : X \to Y$ be a covering map of X onto Y of degree $d \geq 1$ (so each point in Y has d inverse image points in X, counting multiplicities). Let δ be the sum of all branching indices of points of X under f. Then

$$\chi(X) + \delta = d\chi(Y).$$

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$$\chi(X) + \delta = d\chi(Y).$$

If a map is locally like z^n, then its branching index at that point is $n - 1$.

If $X = Y = \overline{\mathbb{C}}$ and f is rational of degree d in the Riemann–Hurwitz formula, we obtain

$$2 + \delta = 2d,$$

so that $\delta = 2d - 2$, as has been mentioned before.

Suppose that X is the disk $B(0,r)$ from which $\overline{n}$ Jordan domains have been defined. Let Y be $\overline{\mathbb{C}} \setminus \cup_{i=1}^{q} \overline{D_i}$. Then $\chi(X) = 1 - \overline{n}$, $\chi(Y) = 2 - q$, and so

$$1 - \overline{n} + \delta = d(2-q),$$

so

$$\overline{n} = (q-2)d + 1 + \delta \geq (q-2)d.$$

The Ahlfors theory of covering surfaces, simply explained.

Let f be a non-constant meromorphic function in $\mathbb{C}$. Consider f in a disk $B(0, r)$. Remove from $B(0, r)$ the inverse image

$$\bigcup_{i=1}^{q} f^{-1}(D_i)$$

to get the set X.

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to get the set X.

It is possible that X is not connected, but to simplify this explanation, let us assume that X is connected.

Recall that

$$X = B(0,r) \setminus \bigcup_{i=1}^{q} f^{-1}(D_i), \quad Y = \overline{\mathbb{C}} \setminus \bigcup_{i=1}^{q} \overline{D_i}.$$

The Ahlfors theory of covering surfaces, simply explained.

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$$X = B(0,r) \setminus \bigcup_{i=1}^{q} f^{-1}(D_i), \quad Y = \overline{\mathbb{C}} \setminus \bigcup_{i=1}^{q} \overline{D_i}.$$

Then f maps X into Y, but due to $f(S(0,r))$, this $f: X \to Y$ is not a covering map, in general.

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A lot of the work in the Ahlfors theory of covering surfaces amounts to estimating the error term caused by $f(S(0,r))$ and showing that this error term is small outside a small exceptional set of values of r.

The Ahlfors theory of covering surfaces, simply explained.

Recall that

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In our simple explanation, we now ignore this error term and only discuss what the main terms are.

The Ahlfors theory of covering surfaces, simply explained.

The average covering number of f in $B(0, r)$ is

$$A(r) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \frac{|f'(te^{i\theta})|^2}{(1 + |f(te^{i\theta})|^2)^2} \, t \, dt \, d\theta.$$

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Now $A(r)$ is usually not an integer, but let us pretend that $f : X \to Y$ is a covering map of degree $A(r)$.

The Ahlfors theory of covering surfaces, simply explained.

Suppose that for $1 \leq i \leq q$, there are $\overline{n}_i$ distinct components of $f^{-1}(D_i)$ completely contained in $B(0, r)$. Then

$$\overline{n} = \sum_{i=1}^{q} \overline{n}_i.$$

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This can now be developed in various ways.

Let n_i is the total number of times that f covers D_i in the union of these components.

Recall that
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Suppose that in each component of $f^{-1}(D_i)$, f covers D_i with multiplicity at least μ_i. Then $n_i \geq \mu_i \overline{n}_i$. Hence

$$\begin{aligned} 2 &\geq \sum_{i=1}^{q} \left(1 - \frac{\overline{n}_i}{A(r)}\right) \\ &\geq \sum_{i=1}^{q} \left(1 - \frac{\overline{n}_i}{n_i} \frac{n_i}{A(r)}\right) \\ &\geq \sum_{i=1}^{q} \left(1 - \frac{1}{\mu_i} \frac{n_i}{A(r)}\right). \end{aligned}$$

The Ahlfors theory of covering surfaces, simply explained.

Suppose that we know that each $n_i \leq A(r)$, which would be true in our simplified model. This gives

$$2 \geq \sum_{i=1}^{q} \left(1 - \frac{1}{\mu_i}\right),$$

which is one of the key conclusions of the Ahlfors theory of covering surfaces.