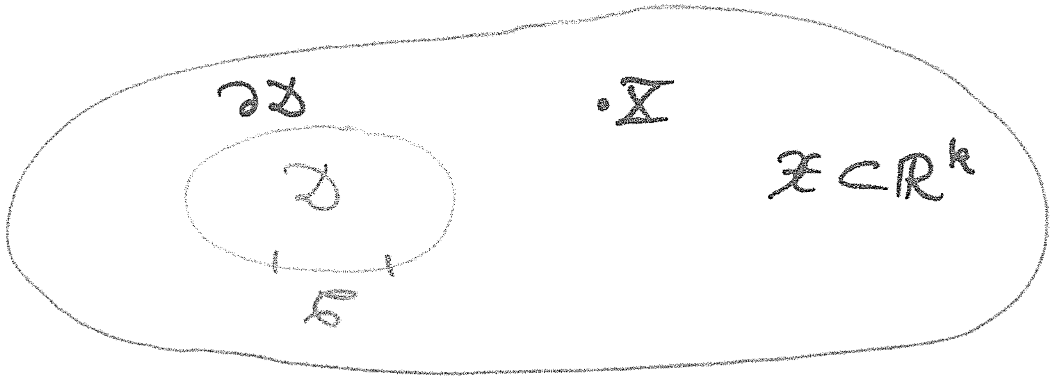
BALANCE LAW



$$\boxed{\mathcal{Q}_{\mathcal{D}}(\partial\mathcal{D}) = \mathcal{P}(\mathcal{D})}$$

PRODUCTION: $\mathcal{P}$ Radon measure

FLUX: $\mathcal{Q}_{\mathcal{D}}$ additive, a.c.

$$\mathcal{Q}_{\mathcal{D}}(\mathcal{E}) = \int_{\mathcal{E}} q_{\mathcal{D}}(X)\, d\mathcal{H}^{k-1}(X)$$

$$\mathcal{Q}_{\mathcal{D}_1}(\mathcal{E}) = \mathcal{Q}_{\mathcal{D}_2}(\mathcal{E})$$

THEOREM Assume:

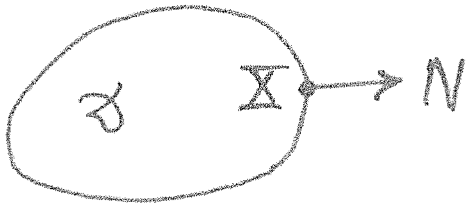
$$\int_{\partial \mathfrak{B}} q_{\mathfrak{B}}(\mathbf{X})\, d\mathcal{H}^{k-1}(\mathbf{X}) = \mathcal{P}(\mathfrak{B})$$

$$|q_{\mathfrak{B}}(\mathbf{X})| < M$$

Then:

(i) $\forall N \in S^{k-1}$, $\exists$ $\alpha_N(\mathbf{X}) \in L^\infty(\mathcal{X})$ such that

$$q_{\mathfrak{B}}(\mathbf{X}) = \alpha_N(\mathbf{X})$$



(ii) $\exists$ $A(\mathbf{X}) \in L^\infty(\mathcal{X}; \mathbb{M}^{1\times k})$ such that

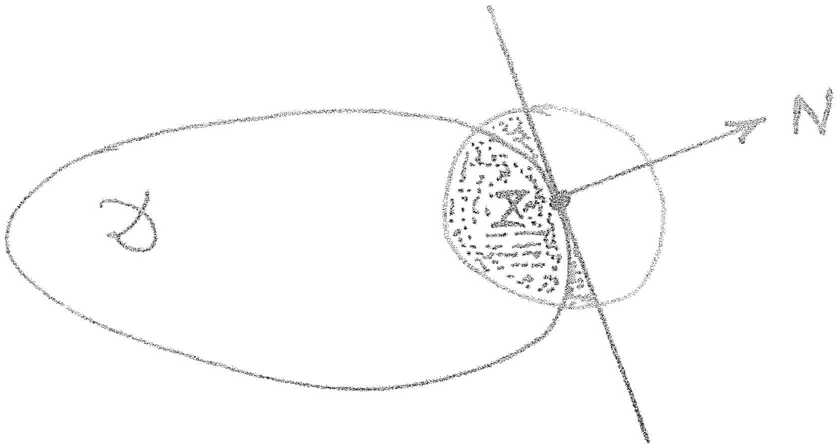
$$\alpha_N(\mathbf{X}) = A(\mathbf{X})N \quad \text{a.e.}$$

(iii) $$\operatorname{div} A = \mathcal{P}$$

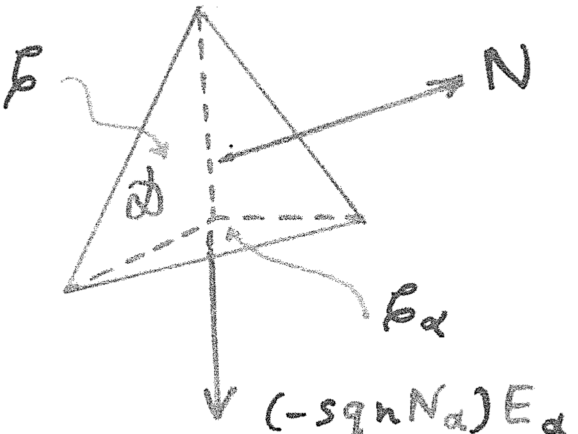
CAUCHY, ...

"PROOF"

(i)



(ii) $$\alpha_{-N}(\mathbf{X}) = -\alpha_N(\mathbf{X})$$



$$\mathcal{H}^{k-1}(\mathcal{E}_\alpha) = |N_\alpha|\,\mathcal{H}^{k-1}(\mathcal{E})$$

$$\int_{\mathcal{E}} \alpha_N \, d\mathcal{H}^{k-1} - \sum_{\alpha=1}^{k} (\operatorname{sgn} N_\alpha) \int_{\mathcal{E}_\alpha} \alpha_{E_\alpha} \, d\mathcal{H}^{k-1} = \mathcal{P}(\mathcal{D})$$

$$\alpha_N(\mathbf{X}) = \sum_{\alpha=1}^{k} \alpha_{E_\alpha}(\mathbf{X}) N_\alpha = A(\mathbf{X})N$$

(iii) $$\int_{\mathcal{D}} \operatorname{div} A_\varepsilon(\mathbf{X})\, d\mathbf{X} = \int_{\partial\mathcal{D}} A_\varepsilon(\mathbf{X}) N(\mathbf{X})\, d\mathcal{H}^{k-1}(\mathbf{X}) = \int_{\mathcal{D}} P_\varepsilon(\mathbf{X})\, d\mathbf{X}$$

CHANGE OF COORDINATES

$\mathbf{X}^* = \mathbf{X}^*(\mathbf{X})$ Lipschitz homeomorphis

$$J = \frac{\partial \mathbf{X}^*}{\partial \mathbf{X}}, \qquad \det J \geq \alpha > 0 \quad \text{a.e.}$$

$$A \in L^1_{loc}(\mathcal{X}; \mathbb{M}^{1\times k}), \quad \mathcal{P} \in \mathcal{M}(\mathcal{X})$$

$$A^* \in L^1_{loc}(\mathcal{X}^*; \mathbb{M}^{1\times k}, \quad \mathcal{P}^* \in \mathcal{M}(\mathcal{X}^*)$$

$$A^* \circ \mathbf{X}^* = (\det J)^{-1} A J^T$$

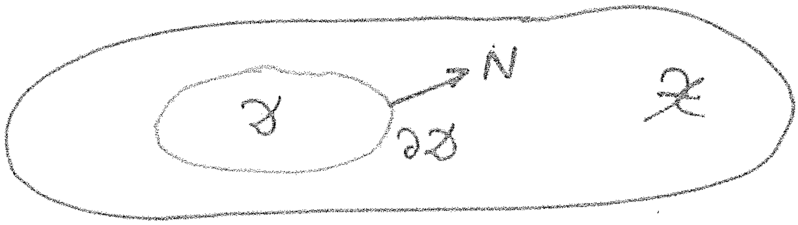
$$\langle \mathcal{P}^*, \varphi^* \rangle = \langle \mathcal{P}, \varphi \rangle, \qquad \varphi = \varphi^* \circ \mathbf{X}^*$$

$$\operatorname{div} A = \mathcal{P} \iff \operatorname{div} A^* = \mathcal{P}^*$$

TRACE THEOREM

$$A \in L^\infty(\mathscr{X}; \mathbb{M}^{1\times k}), \quad \mathcal{P} \in \mathcal{M}(\mathscr{X})$$

$$\operatorname{div} A = \mathcal{P}$$



Then $\exists \; q_{\mathscr{D}} \in L^\infty(\partial\mathscr{D})$ such that

$$\int_{\partial\mathscr{D}} q_{\mathscr{D}}(X)\,\varphi(X)\,d\mathcal{H}^{k-1}(X)$$

$$= \int_{\mathscr{D}} A(X)\,\operatorname{grad}\varphi(X)\,dX + \langle \mathcal{P}, \varphi \rangle$$

for all $\varphi \in C_0^\infty(\mathscr{X})$.

CHEN – FRID

SYSTEMS OF BALANCE LAWS

$$A \in L^\infty(\mathscr{X}; \mathbb{M}^{n \times k}), \quad P \in L^\infty(\mathscr{X}; \mathbb{R}^n)$$

$$\operatorname{div} A = P$$

STATE VECTOR : $\quad U \in \mathbb{R}^n$

"CONSTITUTIVE RELATIONS":

$$A(X) = G(U(X)), \quad P(X) = \Pi(U(X))$$

$$\boxed{\operatorname{div} G(U(X)) = \Pi(U(X))}$$

COMPANION BALANCE LAWS

$Q(U) \in \mathbb{M}^{1\times k}$ is a companion of $G(U) \in \mathbb{M}^{n\times k}$ if $\exists$ $B(U) \in \mathbb{R}^n$ such that

$$DQ_\alpha(U) = B(U)^T D G_\alpha(U), \quad \alpha = 1, \dots, k$$

In that case, any $U(X) \in C^1(\mathcal{X}; \mathbb{R}^n)$ satisfying the system of balance laws

$$\operatorname{div} G(U(X)) = \Pi(U(X))$$

also satisfies the balance law

$$\operatorname{div} Q(U(X)) = h(U(X))$$

$$h(U) = B(U)^T \Pi(U)$$

SYMMETRIC SYSTEMS OF BALANCE LAWS

$$\operatorname{div} G(U(X)) = \Pi(U(X))$$

$$DG_\alpha(U)^T = DG_\alpha(U), \quad \alpha = 1, \dots, k$$

$$G(U)^T = D\Gamma(U)^T$$

COMPANION: $Q(U) = U^T G(U) - \Gamma(U)$

Conversely, if $U^* = B(U)$ is diffeomorphism.

$$G^*(U^*) = G(B^{-1}(U^*))$$

$$\Pi^*(U^*) = \Pi(B^{-1}(U^*))$$

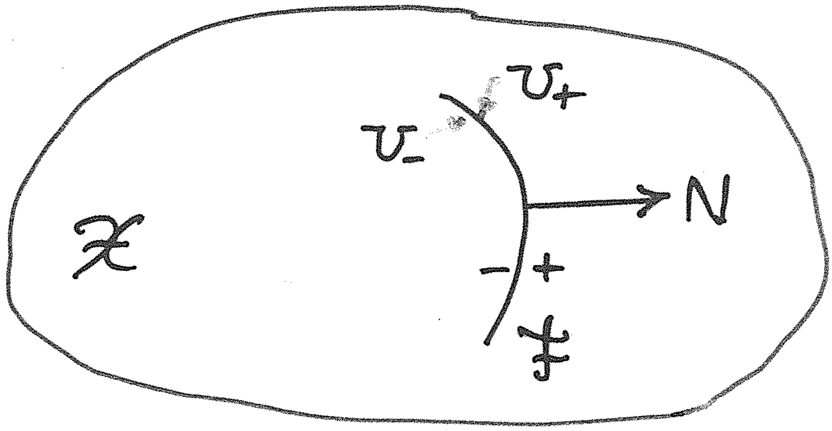
$$Q^*(U^*) = Q(B^{-1}(U^*))$$

$$\Gamma^*(U^*) = U^{*T} G^*(U^*) - Q^*(U^*)$$

Then

$$G^*(U^*)^T = D\Gamma^*(U^*)^T$$

SHOCK FRONTS



$$\operatorname{div} G(U) = \Pi(U)$$

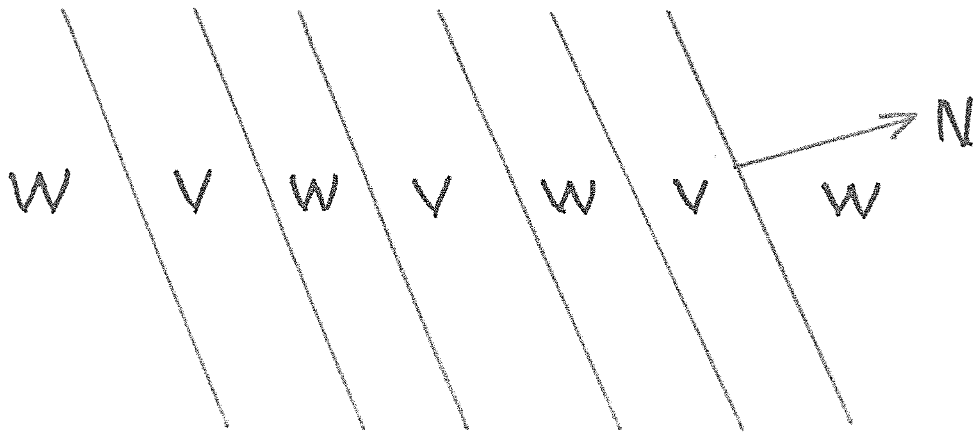
$$[\,G(U_+) - G(U_-)\,]\,N = 0$$

$$\mathcal{V}(U) = \{(N,V) \in S^{k-1} \times \mathbb{R}^n : D[G(U)N]V = 0\}$$

RAPID OSCILLATIONS

$$\operatorname{div} G(U(X)) = 0$$

$$[G(W) - G(V)]N = 0$$

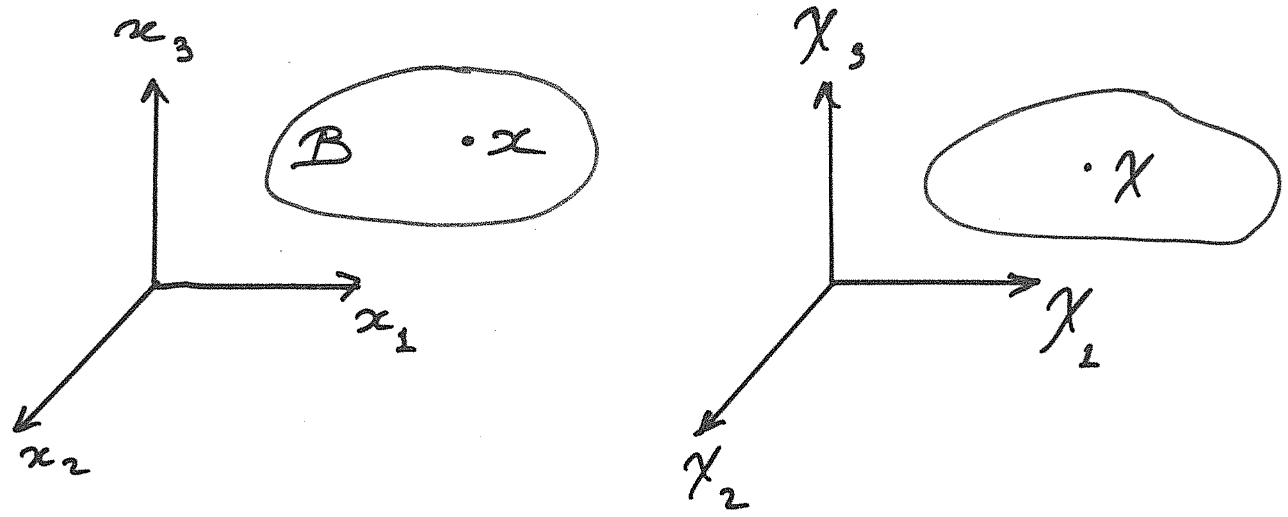


$$U(X) = \rho(X \cdot N)V + [1 - \rho(X \cdot N)]W$$

$$\operatorname{div} Q(U(X)) \leq 0$$

$$[Q(W) - Q(V)]N \neq 0$$

BODIES AND MOTIONS



REFERENCE SPACE PHYSICAL SPACE

reference configuration of body: $\mathcal{B} \subset \mathbb{R}^3$

particle (material point): $x \in \mathcal{B}$

placement: $\chi = \chi(x)$ Lipschitz homeomorphism

motion : $\chi = \chi(x,t)$ Lipschitz continuous

fixed t : $\chi = \chi(\cdot,t)$ placement at time t

motion $\chi = \chi(x,t)$

typical field w

Lagrangian description: $w = f(x,t)$

Eulerian description: $w = \varphi(\chi,t)$

$$\varphi(\chi(x,t),t) = f(x,t)$$

$\dot{w} = \dfrac{\partial f}{\partial t}$ $\qquad w_t = \dfrac{\partial \varphi}{\partial t}$

$\mathrm{Grad}\, w = \mathrm{grad}_x f$ $\qquad \mathrm{grad}\, w = \mathrm{grad}_\chi \varphi$

$\mathrm{Div}\, w = \mathrm{div}_x f$ $\qquad \mathrm{div}\, w = \mathrm{div}_\chi \varphi$

$\nabla w = [\mathrm{Grad}\, w]^T$

VELOCITY: $v = \dot{\chi}$

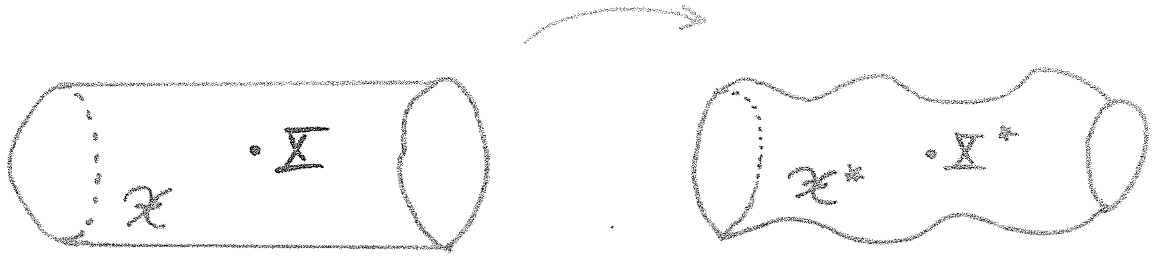
DEFORMATION GRADIENT: $F = \nabla \chi \quad \det F >$

Polar Decomposition: $F = RU$

$U = (F^T F)^{\frac{1}{2}} = U^T, \quad R^T R = R R^T = I$

BALANCE LAWS

motion $y = \chi(x,t)$



$X = (x,t)$ $\qquad$ $X^* = (\chi,t)$

Lagrangian:

$$\boxed{\dot{\Theta} = \mathrm{Div}\,\Psi + P}$$

$$J = \frac{\partial X^*}{\partial X} = \left[\begin{array}{c|c} F & v \\ \hline 0 & 1 \end{array}\right]$$

Eulerian:

$$\boxed{\Theta^*_t + \mathrm{div}(\Theta^* v^T) = \mathrm{div}\,\Psi^* + P^*}$$

$\Theta^* = (\det F)^{-1}\Theta$, $\Psi^* = (\det F)^{-1}\Psi F^T$, $P^* = (\det F$

KINEMATIC BALANCE LAWS

$$F^* = (\det F) F^{-1} = [\partial_F \det F]^T$$

$$\boxed{\begin{aligned} \dot{\overline{\det F}} &= \mathrm{Div}(v^T \partial_F \det F) \\ \dot{F}^* &= \mathrm{Div}(v^T \partial_F F^*) \end{aligned}}$$

$$\Theta^*_t + \mathrm{div}(\Theta^* v^T) = \mathrm{div}(\Psi^*) + P^*$$

$$\Theta^* = 1 \,, \quad \Psi^* = v^T, \quad P^* = 0$$

$$\dot{\Theta} = \mathrm{Div}(\Psi) + P$$

$$\Theta = (\det F)\Theta^* = \det F$$

$$\Psi = (\det F)\Psi^* (F^T)^{-1} = v^T (F^*)^T$$

$$P = (\det F) P^* = 0$$

QIN, DAFERMOS

CONTINUUM THERMOMECHANICS

BALANCE OF MASS:

$$\dot{\rho}_0 = 0$$

$$\rho_t + \mathrm{div}(\rho v^T) = 0$$

$$\rho = \rho_0 (\det F)^{-1}$$

BALANCE OF LINEAR MOMENTUM:

$$(\rho_0 v)^{\bullet} = \mathrm{Div}\, S + \rho_0 b$$

$$(\rho v)_t + \mathrm{div}(\rho v v^T) = \mathrm{div}\, T + \rho b$$

S : Piola stress , T : Cauchy stress

$$T = (\det F)^{-1} S F^T$$

BALANCE OF ANGULAR MOMENTUM:

$$S F^T = F S^T$$

$$T^T = T$$

BALANCE OF ENERGY:

$$(\rho_0 \varepsilon + \tfrac{1}{2}\rho_0 |v|^2)^\circ = \mathrm{Div}(v^T S + Q^T) + \rho_0 v^T b + \rho_0 r$$

ε: internal energy, Q: heat flux, r: source

"BALANCE" OF ENTROPY (CLAUSIUS-DUHEM INEQ.):

$$(\rho_0 s)^\circ \geq \mathrm{Div}\left(\frac{1}{\theta} Q\right) + \rho_0 \frac{r}{\theta}$$

s: (specific) entropy, θ: temperature

DISSIPATION INEQUALITY:

$$\rho_0 \dot{\varepsilon} - \rho_0 \theta \dot{s} - \mathrm{tr}(S \dot{F}^T) - \frac{1}{\theta} Q \cdot G \leq 0$$

$G = \mathrm{Grad}\,\theta$

THERMOELASTICITY

CONSTITUTIVE RELATIONS:

$$\left\{\begin{aligned} \varepsilon &= \hat{\varepsilon}(F, s, G) \\ S &= \hat{S}(F, s, G) \\ \theta &= \hat{\theta}(F, s, G) \\ Q &= \hat{Q}(F, s, G) \end{aligned}\right.$$

DISSIPATION INEQUALITY:

$$\mathrm{tr}[(\rho_0 \hat{\varepsilon}_F - \hat{S})\dot{F}^T] + \rho_0(\hat{\varepsilon}_s - \hat{\theta})\dot{s} + \rho_0 \hat{\varepsilon}_G \dot{G} - \frac{1}{\hat{\theta}}\hat{Q}\cdot G \le 0$$

REDUCTION:

$$\left\{\begin{aligned} \varepsilon &= \hat{\varepsilon}(F, s) \\ S &= \rho_0 \hat{\varepsilon}_F(F, s) \qquad T = \rho \hat{\varepsilon}_F(F,s) F^T \\ \theta &= \hat{\varepsilon}_s(F, s) \\ Q &= \hat{Q}(F, s, G) \end{aligned}\right.$$

$$\hat{Q}(F, s, G)\cdot G \ge 0$$

ADIABATIC-ISENTROPIC THERMOELASTICITY

$$Q \equiv 0 \,, \qquad s \equiv \bar{s} = \text{constant}$$

$$\boxed{\rho_0 \dot{v} = \mathrm{Div}\, S + \rho_0 b}$$

$$(\rho_0 \varepsilon + \tfrac{1}{2}\rho_0 |v|^2)^{\cdot} = \mathrm{Div}(v^T S) + \rho_0 v^T b + \rho_0 r$$

$$\rho_0 \frac{r}{\theta} \leq 0$$

$$\boxed{(\rho_0 \varepsilon + \tfrac{1}{2}\rho_0 |v|^2)^{\cdot} \leq \mathrm{Div}(v^T S) + \rho_0 v^T b}$$

DISSIPATION INEQUALITY

$$\mathrm{tr}[(\rho_0 \hat{\varepsilon}_F - \hat{S})\dot{F}^T] \leq 0$$

$$S = \rho_0 \hat{\varepsilon}_F(F)$$

INCOMPRESSIBILITY

$$\det F = 1 \quad , \quad \rho = \text{constant}$$

DISSIPATION INEQUALITY:

$$\operatorname{tr}[(\rho_0 \hat{\varepsilon}_F - S)\dot{F}^T] \leq 0$$

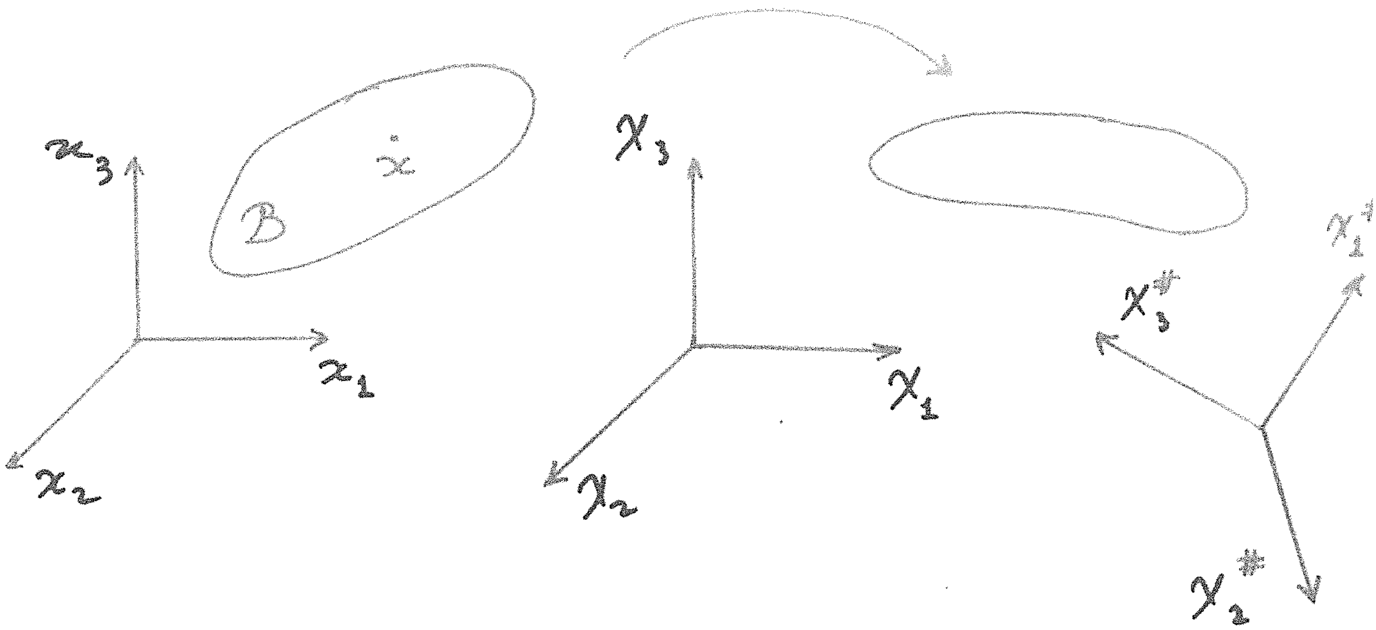
$$0 = \dot{\overline{\det F}} = \operatorname{tr}[(\partial_F \det F)\dot{F}^T = \operatorname{tr}[(F^*)^T \dot{F}^T]$$

$$\operatorname{tr}[(F^{-1})^T \dot{F}^T] = 0$$

$$\boxed{\begin{aligned} S &= -p(F^{-1})^T + \rho_0 \hat{\varepsilon}_F(F) \\ T &= -pI + \rho \hat{\varepsilon}_F(F) F^T \end{aligned}}$$

p: hydrostatic pressure

FRAME INDIFFERENCE



OBSERVER # 1 $\qquad \chi = \chi(x,t)$

OBSERVER # 2 $\qquad \chi^\# = O(t)\,\chi(x,t)$

$$O(t)\,O(t)^T = O(t)^T O(t) = I$$

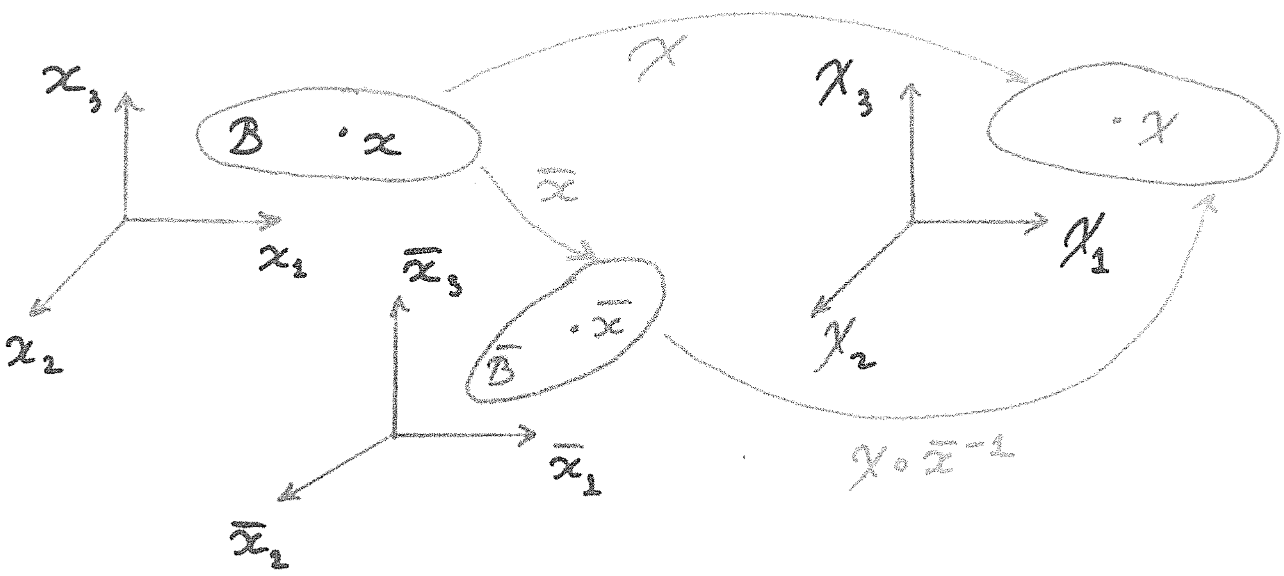
$$\boxed{F^\# = OF, \quad \mathcal{E}^\# = \mathcal{E}, \quad S^\# = OS}$$

$$\hat{\mathcal{E}}(OF) = \hat{\mathcal{E}}(F)$$

POLAR DECOMPOSITION: $F = RU$

$$\hat{\mathcal{E}}(F) = \hat{\mathcal{E}}(U)$$

ISOTROPY GROUP



$$H = \frac{\partial \bar{x}}{\partial x} \in SL(3)$$

$$\bar{F} = F H^{-1}$$

$$\mathcal{E} = \hat{\mathcal{E}}(F) = \bar{\mathcal{E}}(\bar{F})$$

$$\bar{\mathcal{E}}(\bar{F}) = \hat{\mathcal{E}}(\bar{F} H)$$

$$\mathcal{g} = \{ H \in SL(3) : \hat{\mathcal{E}}(F) = \hat{\mathcal{E}}(F H^{-1}) \}$$

FLUID: $\mathcal{G} = SL(3)$

$$\hat{\varepsilon}(F) = \hat{\varepsilon}(FH^{-1}) \quad \forall\ H \in SL(3)$$

$$H = (\det F)^{-1/3} F$$

$$\varepsilon = \tilde{\varepsilon}(\det F) = \breve{\varepsilon}(\rho)$$

$$T = -p(\rho) I, \quad p = \rho^2 \breve{\varepsilon}_\rho(\rho)$$

ISOTROPIC SOLID: $\mathcal{G} = SO(3)$

$$\hat{\varepsilon}(F) = \hat{\varepsilon}(FO) \quad \forall\ O \in SO(3)$$

$$\hat{\varepsilon}(U) = \hat{\varepsilon}(O^T U O) \quad \forall\ O \in SO(3)$$

$$\varepsilon = \tilde{\varepsilon}(J_1, J_2, J_3)$$

J_1, J_2, J_3 principal invariants of U

THERMOVISCOELASTICITY

CONSTITUTIVE RELATIONS:

$$
\begin{cases}
\varepsilon = \hat{\varepsilon}(F, \dot{F}, s, G) \\
S = \hat{S}(F, \dot{F}, s, G) \\
\theta = \hat{\theta}(F, \dot{F}, s, G) \\
Q = \hat{Q}(F, \dot{F}, s, G)
\end{cases}
$$

DISSIPATION INEQUALITY:

$$
\mathrm{tr}[(\rho_0 \hat{\varepsilon}_F - \hat{S})\dot{F}^T] + \mathrm{tr}[\rho_0 \hat{\varepsilon}_{\dot{F}} \ddot{F}^T] + \rho_0(\hat{\varepsilon}_s - \hat{\theta})\dot{s} + \rho_0 \hat{\varepsilon}_G \dot{G} - \frac{\hat{Q} \cdot G}{\hat{\theta}} \leq 0
$$

REDUCTION:

$$
\begin{cases}
\varepsilon = \hat{\varepsilon}(F, s) \\
S = \rho_0 \hat{\varepsilon}_F(F, s) + Z(F, \dot{F}, s, G) \\
\theta = \hat{\varepsilon}_s(F, s) \\
Q = \hat{Q}(F, \dot{F}, s, G)
\end{cases}
$$

$$
\mathrm{tr}[Z(F, \dot{F}, s, G)\dot{F}^T] - \frac{1}{\hat{\theta}} \hat{Q}(F, \dot{F}, s, G) \cdot G \leq 0
$$

ADIABATIC – ISENTROPIC THERMOVISCOELASTICITY

$Q \equiv 0 \quad , \quad s = \bar{s} = \text{constant}$

$$\begin{cases} \varepsilon = \hat{\varepsilon}(F) \\ S = \rho_0 \hat{\varepsilon}_F(F) + Z(F, \dot{F}) \end{cases}$$

$$\operatorname{tr}[Z(F,\dot{F})\dot{F}^T] \leq 0$$

$$\boxed{\rho_0 \dot{v} = \operatorname{Div} S + \rho_0 b}$$

$$(\rho_0 \varepsilon + \tfrac{1}{2}\rho_0 |v|^2)^{\cdot} = \operatorname{Div}(v^T S) + \rho_0 v^T b + \rho_0 r$$

$$\rho_0 \frac{r}{\theta} \leq 0$$

$$\boxed{(\rho_0 \varepsilon + \tfrac{1}{2}\rho_0 |v|^2)^{\cdot} \leq \operatorname{Div}(v^T S) + \rho_0 v^T b}$$

REDUCTION OF $Z(F,\dot{F})$

$$S = \rho_0 \hat{\varepsilon}_F(F) + Z(F,\dot{F})$$

$$\dot{F}_{i\alpha} = \frac{\partial v_i}{\partial X_\alpha} \quad , \quad L_{ij} = \frac{\partial v_i}{\partial x_j}$$

$$\dot{F} = LF$$

$$Z(F,\dot{F}) = \hat{Z}(F,L)$$

$$L = D + W \quad , \quad D = \tfrac{1}{2}(L+L^T) \, , \, W = \tfrac{1}{2}(L-L^T)$$

FRAME INDIFFERENCE: $F \sim O(t)F \quad , \; O(0) = I$

$$\hat{Z}(OF, OLO^T + \dot{O}O^T) = O\hat{Z}(F,L)$$

$$\hat{Z}(F,L) = \hat{Z}(F,D)$$

$$\hat{Z}(OF, ODO^T) = O\hat{Z}(F,D)$$

VISCOELASTIC FLUID

PIOLA-KIRCHHOFF STRESS: $S = \rho_0 \hat{\varepsilon}_F(F) + \hat{Z}(F, D)$

CAUCHY STRESS: $T = (\det F)^{-1} S F^T = \rho \hat{\varepsilon}_F(F) F^T + \tilde{Z}(F, D)$

$$\tilde{Z}(OF, ODO^T) = O \tilde{Z}(F, D) O^T$$

FLUID $\Leftrightarrow$ ISOTROPY GROUP IS $SL(3)$

$$\hat{\varepsilon}(FH) = \hat{\varepsilon}(F), \quad \tilde{Z}(FH, D) = \tilde{Z}(F, D)$$

$$\hat{\varepsilon}(F) = \bar{\varepsilon}(\rho), \quad \tilde{Z}(F, D) = \bar{Z}(\rho, D)$$

$$\bar{Z}(\rho, ODO^T) = O \bar{Z}(\rho, D) O^T$$

$$\boxed{T = -pI + A_0 I + A_1 D + A_2 D^2}$$

$p = p(\rho) = \rho^2 \bar{\varepsilon}_\rho(\rho)$

A_i : functions of ρ and the principal invariants of D

NEWTONIAN FLUID: $\boxed{T = -p(\rho) I + \lambda(\rho)(\mathrm{tr} D) I + 2\mu(\rho) D}$

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0$$

μ : SHEAR VISCOSITY $\quad \lambda + \frac{2}{3}\mu$: BULK VISCOSITY

HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

$$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) = 0$$

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0$$

$$x \in \mathbb{R}^m, \quad U \in \mathbb{R}^n$$

HYPERBOLIC: For any $\nu \in S^{m-1}$, $U \in \mathbb{R}^n$,

$$\Lambda(\nu; U) = \sum_{\alpha=1}^{m} \nu_\alpha DG_\alpha(U)$$

has real eigenvalues $\lambda_1(\nu; U), \dots, \lambda_n(\nu; U)$ and n linearly independent eigenvectors $R_1(\nu; U), \dots, R_n(\nu; U)$

INTERPRETATION OF HYPERBOLICITY

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

Linearize about state U:

$$\partial_t V + \sum_{\alpha=1}^{m} DG_\alpha(U)\,\partial_\alpha V = 0$$

Hyperbolicity $\Leftrightarrow$ Existence of wave-like solutions:

$$V(x,t) = \exp[\nu\cdot x - \lambda_j(\nu;U)\,t]\,R_j(\nu;U)$$

Shock waves for the nonlinear system propagating in direction ν with speed s: <u>Rankine-Hugoniot jump condition</u>:

$$s\,[U_+ - U_-] = \nu\cdot[G(U_+) - G(U_-)]$$

weak shock: $s \sim \lambda_j$, $U_+ - U_- \sim R_j$

EULER EQUATIONS

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v^T) = 0 \\ \partial_t (\rho v) + \operatorname{div}(\rho v v^T) + \operatorname{grad} p(\rho) = 0 \end{cases}$$

$$p(\rho) = \rho^2 \varepsilon_\rho(\rho)$$

HYPERBOLIC: $p_\rho(\rho) > 0$

ISENTROPIC ELASTODYNAMICS

$$U = (F, v)\ ,\quad F \in \mathbb{M}^{3\times 3}\ ,\quad v \in \mathbb{R}^3\ ,\quad \rho_0 \equiv 1$$

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0 & i, \alpha = 1, 2, 3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F) = 0 & i = 1, 2, 3 \end{cases}$$

$$S_{i\alpha}(F) = \frac{\partial \varepsilon(F)}{\partial F_{i\alpha}}$$

HYPERBOLIC : $\varepsilon(F)$ rank-one convex

$$\frac{\partial^2 \varepsilon(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \nu_\alpha \nu_\beta \xi_i \xi_j > 0, \qquad \forall \nu, \xi \in S^2$$

F : deformation gradient

v : velocity

S : Piola stress tensor

ε : internal (strain) energy

BORN-INFELD

$$U = (B, D), \quad B \in \mathbb{R}^3, \quad D \in \mathbb{R}^3$$

MAXWELL'S EQS:
$$\begin{cases} \partial_t B = -\operatorname{curl} E \\ \partial_t D = \operatorname{curl} H \end{cases}$$

$$\eta = \left[1 + |B|^2 + |D|^2 + |Q|^2\right]^{1/2}, \quad Q = D \wedge B$$

$$\begin{cases} E = \dfrac{\partial \eta}{\partial D} = \dfrac{1}{\eta}\left[D + B \wedge Q\right] \\ H = \dfrac{\partial \eta}{\partial B} = \dfrac{1}{\eta}\left[B - D \wedge Q\right] \end{cases}$$

E : electric field

H : magnetic field

D : electric displacement

B : magnetic induction

BRENIER

ENTROPY – ENTROPY FLUX PAIRS

$$\partial_t U + \operatorname{div} G(U) = 0$$

$\eta(U)$ entropy, $Q(U)$ entropy flux if

$$DQ_\alpha(U) = D\eta(U)\,DG_\alpha(U), \quad \alpha = 1, \dots, m$$

U classical solution $\Rightarrow \partial_t \eta(U) + \operatorname{div} Q(U) =$

$$D^2\eta(U)\,DG_\alpha(U) = DG_\alpha(U)^T D^2\eta(U)$$

SYMMETRIC: $\quad DG_\alpha(U)^T = DG_\alpha(U), \quad \alpha = 1, \dots,$

$$\eta(U) = \tfrac{1}{2}|U|^2$$

CONVERSELY: $\quad U^* = D\eta(U)$ symmetrizes

GODUNOV. KRUZKOV. LAX. FRIEDRICHS. ...

EULER EQUATIONS

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0 \\ \partial_t(\rho v) + \mathrm{div}(\rho v v^T) + \mathrm{grad}\, p(\rho) = 0 \end{cases}$$

ENTROPY – ENTROPY FLUX PAIR: $(p(\rho) = \rho^2 \varepsilon_\rho(\rho$

$$\begin{cases} \eta = \rho\varepsilon(\rho) + \frac{1}{2}\rho|v|^2 \\ Q = [\rho\varepsilon(\rho) + \frac{1}{2}\rho|v|^2 + p(\rho)]\, v^T \end{cases}$$

$\eta(\rho, v)$ CONVEX $\Leftrightarrow$ $p_\rho(\rho) > 0$

ISENTROPIC ELASTODYNAMICS

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i,\alpha = 1,2,3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F) = 0, & i = 1,2,3 \end{cases}$$

$$S_{i\alpha}(F) = \frac{\partial \varepsilon(F)}{\partial F_{i\alpha}}$$

ENTROPY-ENTROPY FLUX PAIR:

$$\begin{cases} \eta = \varepsilon(F) + \frac{1}{2}|v|^2 \\ Q_\alpha = -v_i S_{i\alpha}(F) \end{cases}$$

$\varepsilon(F)$ nonconvex !

FRAME INDIFFERENCE: $\varepsilon = \varepsilon(U)$

$\varepsilon(F) \nearrow \infty$ as $\det F \searrow 0$

BORN-INFELD

$$\begin{cases} \partial_t B = -\operatorname{curl} E \\ \partial_t D = \operatorname{curl} H \end{cases}$$

$$E = \frac{\partial \eta}{\partial D}, \quad H = \frac{\partial \eta}{\partial B}$$

$$\eta = [1 + |B|^2 + |D|^2 + |Q|^2]^{1/2}, \quad Q = D \wedge B$$

ENTROPY-ENTROPY FLUX PAIR: (η, Q)

$\eta(B, D)$ nonconvex!

CONVEX ENTROPY $\Rightarrow$ CAUCHY PROBLEM WELL-POSE

$$\begin{cases} \partial_t U(x,t) + \operatorname{div} G(U(x,t)) = 0, & x \in \mathbb{R}^m, \ t > 0 \\ U(x,0) = U_0(x), & x \in \mathbb{R}^m \end{cases}$$

$\eta(U)$ entropy, $D^2\eta(U)$ positive definite

<u>THEOREM</u> If $\nabla U_0 \in H^\ell(\mathbb{R}^m)$, $\ell > \frac{m}{2}$, then $\exists$ unique C^1 solution U on maximal time interval $[0,T)$:

$$\nabla U(\cdot,t) \in C^0([0,T); H^\ell(\mathbb{R}^m)).$$

If $T < \infty$, then

$$\int_0^T \|\nabla U(\cdot,t)\|_{L^\infty} \, dt = \infty$$

SCHAUDER, ..., KATO, MAJDA, ..

METHODS OF PROVING EXISTENCE

I. Via linearization and fixed point

For given V solve Cauchy problem

$$\begin{cases} \partial_t U + \sum_{\alpha=1}^{m} DG_\alpha(V)\,\partial_\alpha U = 0 \\ U(x,0) = U_0(x) \end{cases}$$

Show that the map $V \mapsto U$ has a fixed point

Basic estimate:

$$2U^T D^2\eta(V)\,\partial_t U + \sum_{\alpha=1}^{m} 2U^T D^2\eta(V)\,DG_\alpha(V)\,\partial_\alpha U = 0$$

$$\partial_t[U^T D^2\eta(V)\,U] + \sum_{\alpha=1}^{m} \partial_\alpha[U^T D^2\eta(V)\,DG_\alpha(V)\,U] = \text{Error}$$

II. Via vanishing viscosity and compactness

For $\varepsilon > 0$, solve the Cauchy problem

$$\begin{cases} \partial_t U_\varepsilon + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U_\varepsilon) = \varepsilon \Delta U_\varepsilon \\ U_\varepsilon(x,0) = U_0(x) \end{cases}$$

Show that as $\varepsilon \to 0$, $U_\varepsilon \to U$

Essentially same estimates

EULER EQUATIONS WITH VACUUM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v v^T) + \operatorname{grad} p(\rho) = 0 \end{cases}$$

$$p(\rho) = \frac{1}{\gamma}\rho^\gamma, \quad \varepsilon(\rho) = \frac{1}{\gamma(\gamma-1)}\rho^{\gamma-1}, \quad \gamma \in (1, 5/3)$$

$$\eta(\rho, v) = \frac{1}{\gamma(\gamma-1)}\rho^\gamma + \frac{1}{2}\rho v^2$$

$$\begin{cases} \partial_t \rho + (v \cdot \operatorname{grad})\rho + \rho \operatorname{div} v \\ \partial_t v + (v \cdot \operatorname{grad}) v + \rho^{\gamma-1} \operatorname{grad} \rho = 0 \end{cases}$$

hyperbolic for $\rho > 0$, but not for $\rho = 0$!

new state variables (w, v), $w = \frac{2}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}$

$$\begin{cases} \partial_t w + (v \cdot \operatorname{grad}) w + \frac{\gamma-1}{2} w \operatorname{div} v = 0 \\ \partial_t v + (v \cdot \operatorname{grad}) v + \frac{\gamma-1}{2} w \operatorname{grad} w = 0 \end{cases}$$

Above system symmetric and hence hyperbolic even at $w = 0$ (i.e. $\rho = 0$)

$$\eta(w, v) = \frac{1}{2}(w^2 + v^2)$$

BLOWING UP OF CLASSICAL SOLUTIONS

Burgers equation:

$$\begin{cases} \partial_t u + \partial_x \left(\tfrac{1}{2}u^2\right) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

characteristics:

$$\frac{dx}{dt} = u(x,t)$$

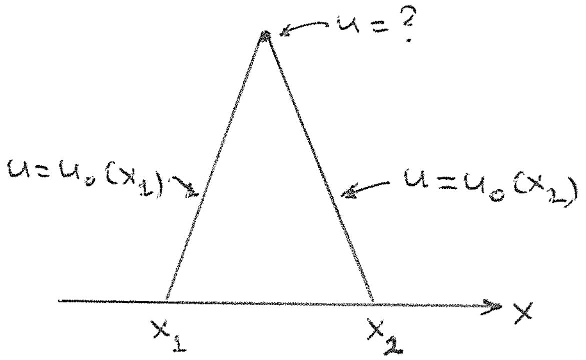
derivative in characteristic direction:

$$\frac{d}{dt} = \partial_t + u\,\partial_x$$

$\frac{du}{dt} = 0 \Rightarrow$ u = constant along characteristics

$\Rightarrow$ characteristics straight lines with slope u

$u_0(x_1) > u_0(x_2) \Rightarrow$



CHALLIS (1848)

WAVE BREAKING

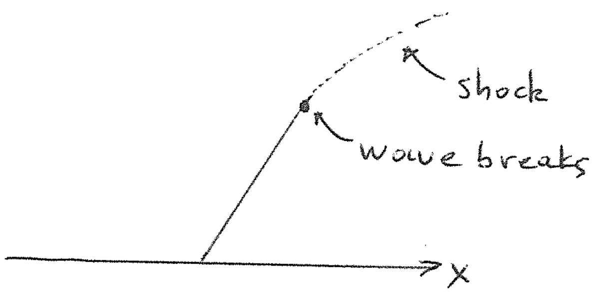
$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2} u^2\right) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

$$\frac{d}{dt} = \partial_t + u\,\partial_x$$

$v = \partial_x u$, $\quad v_0 = \partial_x u_0$, $\quad \bar{v}_0 = \min v_0 < 0$

$$\frac{dv}{dt} + v^2 = 0$$

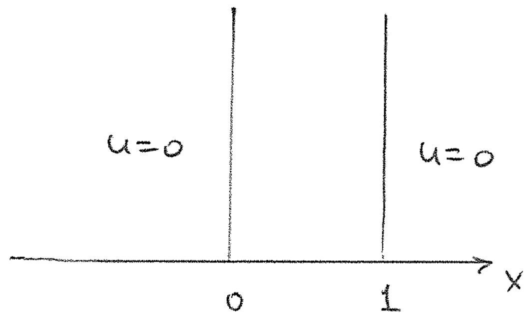
wave breaks at $t = -\frac{1}{\bar{v}_0}$



STOKES (1848)

ALTERNATIVE SCENARIO

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad \operatorname{spt} u_0 \subset (0,1)$$



$$M(t) = \int_0^1 x\, u(x,t)\, dx$$

$$M^2(t) \leq \int_0^1 x^2 dx \int_0^1 u^2(x,t)\, dx = \frac{1}{3}\int_0^1 u^2(x,t)\, dx$$

$$\dot{M}(t) = -\frac{1}{2}\int_0^1 x\, \partial_x [u^2(x,t)]\, dx = \frac{1}{2}\int_0^1 u^2(x,t)\, dx \geq \frac{3}{2} M^2(t)$$

$M(0) > 0 \Rightarrow$ life span of classical solution $\leq \frac{2}{3} M(0)^{-1}$

life span of solutions before waves break $\leq \frac{1}{6} M(0)^{-1}$

Above scenario never realized.

ADMISSIBLE WEAK SOLUTIONS

$$\partial_t U + \operatorname{div} G(U) = 0$$

WEAK SOLUTION: $U \in L^\infty$, satisfies system in sense of distributions

NONUNIQUENESS!

ENTROPY-ENTROPY FLUX PAIR: (η, Q)

ADMISSIBILITY CONDITION:

$$\partial_t \eta(U) + \operatorname{div} Q(U) \leq 0,$$

in the sense of distributions

REMARKS

- Is $v \in L^\infty(\mathbb{R}^m \times [0,T))$ a natural assumption?

- $v \in L^\infty(\mathbb{R}^m \times [0,T)) \Rightarrow v \in C_{weak}([0,T); L^\infty(\mathbb{R}^m))$

- NORMALIZATION: $\eta(0) = 0, \; D\eta(0) = 0$

- ASSUMPTION: $v \in L^\infty([0,T); L^2(\mathbb{R}^m))$

Hence: $\int_{\mathbb{R}^m} \eta(v(\cdot,t))\,dx$ bounded.

- $$\partial_t \eta(v) + \operatorname{div} Q(v) = P$$

  P is a measure

SOME CONSEQUENCES OF ADMISSIBILITY

Under assumption $D^2\eta(U)$ positive definite:

For an at most countable $J \subset [0,T)$, $\tau \in [0,T) \setminus J$ iff

- $t \mapsto U(\cdot,t)$ strongly continuous in $L^p(\mathbb{R}^m)$, $p \in [1,\infty)$, at $t=\tau$
- $$\int_\tau^\infty \int_{\mathbb{R}^m} [\partial_t \psi \, \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(U)] \, dx\, dt + \int_{\mathbb{R}^m} \psi(x,\tau)\, \eta(U(x,\tau))\, dx \geq 0$$
- $$\int_{\mathbb{R}^m} \eta(U(x,t))\, dx \leq \int_{\mathbb{R}^m} \eta(U(x,\tau))\, dx \,, \quad t \geq \tau$$

Corollary: $J = \emptyset$ iff $t \mapsto \int_{\mathbb{R}^m} \eta(U(x,t))\, dx$ decreasing on $[0,T$

ALL OF THE ABOVE VALID EVEN IF $\eta(U)$ IS NOT CONVEX

PROVIDED THAT $U \mapsto \int \eta(U)\, dx$ l.s.c. IN L^∞ wk*

ELASTODYNAMICS

$$\eta = \varepsilon(F) + \tfrac{1}{2}|v|^2$$

$F \mapsto \int \varepsilon(F)\,dx$ l.s.c. in L^∞ wk* $\iff$ ε quasiconvex

- K unit cube in $\mathbb{R}^m$
- $\bar{F}$ constant matrix, $\det \bar{F} > 0$
- $\chi \in W^{1,\infty}(K)$, $\chi(x) = \bar{F}x$ for $x \in \partial K$, $F = \operatorname{Grad}\chi$

$\Longrightarrow$ $$\varepsilon(\bar{F}) \le \int_K \varepsilon(F)\,dx$$

QUASICONVEXITY $\Longrightarrow$ RANK ONE CONVEXITY

$\not\Longleftarrow$ SVERAK

POLYCONVEXITY: $\varepsilon(F) = g(F, F^*, \det F)$, g convex

POLYCONVEXITY $\Longrightarrow$ QUASICONVEXITY

because $F \mapsto (F, F^*, \det F)$ continuous in L^∞ wk* !!!

CONVEX ENTROPY $\Rightarrow$ L^2 STABILITY OF C^1 SOLUTIONS

$$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) = 0\,, \quad x \in \mathbb{R}^m,\ t>0$$

$\eta(U)$ entropy, $D^2\eta(U)$ positive definite

THEOREM Assume:

$\bar{U}$: C^1 solution on $[0,T)$

U: admissible L^∞ solution on $[0,T)$

Then, for any $r>0$ and $t \in (0,T)$,

$$\int_{|x|<r} |U(x,t) - \bar{U}(x,t)|^2 dx \le a e^{bt} \int_{|x|<r+st} |U(x,0) - \bar{U}(x,0)|^2 dx$$

DAFERMOS, DIPERNA

"PROOF"

$$\partial_t \bar{U} + \operatorname{div} G(\bar{U}) = 0$$

$$\partial_t U + \operatorname{div} G(U) = 0$$

$$\partial_t \eta(\bar{U}) + \operatorname{div} Q(\bar{U}) = 0$$

$$\partial_t \eta(U) + \operatorname{div} Q(U) \leq 0$$

$$\underline{h(U,\bar{U}) = \eta(U) - \eta(\bar{U}) - D\eta(\bar{U})[U-\bar{U}]}$$

$$Y(U,\bar{U}) = Q(U) - Q(\bar{U}) - D\eta(\bar{U})[G(U) - G(\bar{U})]$$

$$Z_\alpha(U,\bar{U}) = D^2\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U}) - DG_\alpha(\bar{U})[U-\bar{U}]]$$

$$\partial_t h(U,\bar{U}) + \operatorname{div} Y(U,\bar{U})$$

$$\leq -\partial_t \bar{U}^T D^2\eta(\bar{U})[U-\bar{U}] - \partial_\alpha \bar{U}^T D^2\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})]$$

$$= \partial_\alpha \bar{U}^T DG_\alpha(\bar{U})^T D^2\eta(\bar{U})[U-\bar{U}] - \partial_\alpha \bar{U}^T D^2\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U}$$

$$= \partial_\alpha \bar{U}^T D^2\eta(\bar{U}) DG_\alpha(\bar{U})[U-\bar{U}] - \partial_\alpha \bar{U}^T D^2\eta(\bar{U})[G_\alpha(U) - G_\alpha($$

$$\underline{= -\partial_\alpha \bar{U}^T Z_\alpha(U,\bar{U})}$$

INVOLUTIONS

ELASTODYNAMICS:

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F) = 0 \end{cases}$$

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0$$

BORN-INFELD:

$$\begin{cases} \partial_t B = -\operatorname{curl} E \\ \partial_t D = \operatorname{curl} H \end{cases}$$

$$\operatorname{div} B = 0, \quad \operatorname{div} D = 0$$

BOILLAT, DAFERMOS

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

$k \times n$

$$M_\alpha G_\beta(U) + M_\beta G_\alpha(U) = 0$$

INVOLUTION: $\displaystyle\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U = 0$

INVOLUTION CONE: $\displaystyle\mathcal{C} = \bigcup_{\nu \in S^{m-1}} \ker N(\nu)$

$$N(\nu) = \sum_{\alpha=1}^{m} \nu_\alpha M_\alpha$$

$$\Lambda(\nu; U) = \sum_{\alpha=1}^{m} \nu_\alpha DG_\alpha(U)$$

$$N(\nu)\Lambda(\nu; U) = 0$$

$$\operatorname{rank} N(\nu) = \dim \ker \Lambda(\nu; U)$$

CAUCHY PROBLEM — CLASSICAL SOLUTIONS

$$\begin{cases} \partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, & x \in \mathbb{R}^m, t > 0 \\ U(x,0) = U_0(x), & x \in \mathbb{R}^m \end{cases}$$

INVOLUTION: $\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U(x,t) = 0$

$\eta(U)$ entropy, convex in direction of $\mathcal{C}$

THEOREM If $\nabla U_0 \in H^\ell(\mathbb{R}^m)$, $\ell > \frac{m}{2}$, then $\exists$ unique C^1 solution U on maximal time interval $[0,T)$:

$$\nabla U(\cdot,t) \in C^0([0,T); H^\ell(\mathbb{R}^m)).$$

If $T < \infty$, then

$$\int_0^T \|\nabla U(\cdot,t)\|_{L^\infty} dt = \infty$$

L^2 STABILITY OF C^1 SOLUTIONS

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, \quad x \in \mathbb{R}^m, \ t > 0$$

INVOLUTION : $\displaystyle\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U(x,t) = 0$

$\eta(U)$ entropy, convex in direction of $\mathscr{C}$

THEOREM Assume

$\bar{U}$: C^1 solution on $[0,T)$

U : admissible L^∞ solution on $[0,T)$

$$\limsup_{\varepsilon \downarrow 0} \sup_{|y-x|<\varepsilon} |U(y,t) - U(x,t)| << 1$$

Then, for any $t \in [0,T)$,

$$\int_{\mathbb{R}^m} |U(x,t) - \bar{U}(x,t)|^2 dx \leq a(t) \int_{\mathbb{R}^m} |U(x,0) - \bar{U}(x,0)|^2 dx$$

INVOLUTION CONE IN EXAMPLES

ELASTODYNAMICS:

$$\mathcal{C} = \{(F, v) : F = \xi \otimes \nu, \ \xi, \nu, v \in \mathbb{R}^3\}$$

entropy convex in direction of $\mathcal{C}$ if and only if $\mathcal{E}(F)$ is rank-one convex

BORN-INFELD

$$\mathcal{C} = \mathbb{R}^6$$

entropy is nonconvex in direction of $\mathcal{C}$

EXTRA ENTROPIES

ELASTODYNAMICS:

$$\partial_t (\det F) = \mathrm{div}(v^T \partial_F \det F)$$

$$\partial_t F^* = \mathrm{div}(v^T \partial_F F^*)$$

BORN – INFELD:

$$Q = D \wedge B \quad \text{(Poynting vector)}$$

$$\partial_t Q = \mathrm{div}\left[\frac{1}{\eta}(I + BB^T + DD^T - QQ^T)\right]$$

SERRE, BRENIER

CONTINGENT ENTROPIES

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

INVOLUTION: $\displaystyle\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U = 0$

(η, Q) is a contingent entropy-entropy flux pair if

$$DQ_\alpha(U) = D\eta(U)\, DG_\alpha(U) + \underbrace{\Xi(U)^T}_{\in \mathbb{R}^k} M_\alpha\,, \quad \alpha = 1, \dots, m$$

If U is a C^1 solution satisfying the involution:

$$\partial_t \eta(U) + \sum_{\alpha=1}^{m} \partial_\alpha Q_\alpha(U) = 0$$

POLYCONVEX ENTROPIES

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

INVOLUTION: $\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U = 0$

(η, Q) primary (contingent) entropy-entropy flux pair

$(\Phi_1, \Psi_1), \dots, (\Phi_N, \Psi_N)$ auxiliary contingen entropy-entropy flux pairs, including $(U^i, G^i$ arranged into (Φ, Ψ), $\Phi \in \mathbb{R}^N$, $\Psi \in \mathbb{M}^{N\times m}$

η is polyconvex if

$\eta(U) = \theta(\Phi(U))$, with θ convex on $\mathbb{R}^N$

ELASTODYNAMICS:

primary entropy-entropy flux pair:

$$\eta = \varepsilon(F) + \tfrac{1}{2}|v|^2, \quad Q_\alpha = -v_i S_{i\alpha}(F)$$

auxiliary entropies:

$$\Phi = (F, F^*, \det F, v)$$

η polyconvex if

$$\varepsilon(F) = \theta(F, F^*, \det F)$$

where θ is convex on $\mathbb{R}^{19}$

BORN–INFELD:

primary entropy:

$$\eta = [1 + |B|^2 + |D|^2 + |Q|^2]^{1/2}$$

auxiliary entropies:

$$\Phi = (B, D, Q)$$

η is polyconvex

CAUCHY PROBLEM – CLASSICAL SOLUTIONS

$$\begin{cases} \partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, & x \in \mathbb{R}^m,\ t>0 \\ U(x,0) = U_0(x), & x \in \mathbb{R}^m \end{cases}$$

INVOLUTION: $\displaystyle\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U(x,t) = 0$

$\eta(U)$ polyconvex (contingent) entropy

THEOREM If $\nabla U_0 \in H^\ell(\mathbb{R}^m)$, $\ell > \frac{m}{2}$, then $\exists$ unique C^1 solution U on maximal time interval $[0,T)$:

$$\nabla U(\cdot,t) \in C^0([0,T); H^\ell(\mathbb{R}^m))$$

If $T<\infty$, then

$$\int_0^T \|\nabla U(\cdot,t)\|_{L^\infty}\, dt = \infty$$

L^2 STABILITY OF C^1 SOLUTIONS

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, \quad x \in \mathbb{R}^m, \; t > 0$$

INVOLUTION: $\displaystyle\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U(x,t) = 0$

$\eta(U)$ polyconvex (contingent) entropy

THEOREM Assume:

$\bar{U}$: C^1 solution on $[0,T)$

U : admissible L^∞ solution on $[0,T)$

Then, for any $r>0$ and $t \in (0,T)$,

$$\int_{|x|<r} |U(x,t) - \bar{U}(x,t)|^2 dx \le a e^{bt} \int_{|x|<r+st} |U(x,0) - \bar{U}(x,0)|^2 dx$$

EXTENDED SYSTEMS

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

INVOLUTION: $$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U = 0$$

(Φ, Ψ) auxiliary contingent entropy-entropy flux pairs, $\Phi \in \mathbb{R}^N$, $\Psi \in \mathbb{M}^{N\times m}$

(η, Q) primary contingent entropy-entropy flux pair, $\eta(U) = \theta(\Phi(U))$ polyconvex

Try to find functions S and Π defined on $\mathbb{R}^N$ and taking values in $\mathbb{M}^{N\times m}$ and $\mathbb{M}^{1\times m}$, respectively, such that

$$S(\Phi(U)) = \Psi(U), \quad \Pi(\Phi(U)) = Q(U)$$

and, in addition, $(\theta(\underline{X}), \Pi(\underline{X}))$ is an entropy-entropy flux pair for the system

$$\partial_t \underline{X} + \sum_{\alpha=1}^{m} \partial_\alpha S_\alpha(\underline{X}) = 0 .$$

Then solve above system with initial data

$$\underline{X}(x,0) = \Phi(U_0(x)) .$$

For this approach see

ELASTODYNAMICS : DEMOULINI-STUART-TZAVARAS

BORN-INFELD : BRENIER, SERRE