Non-uniqueness for the incompressible Euler equations up to Onsager's critical exponent

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Dissipation of energy and uniqueness

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- C^1 solutions: dissipation of energy implies uniqueness among C^1 solutions (straightforward) and among the dissipative weak solutions with the same initial data [Lions; Brenier, De Lellis and Székelyhidi]
- L^∞ solutions: Let $e \in C([0,T];\mathbb{R}^+)$. Then, there exist initial data $v_0 \in L^\infty$ having infinitely many weak solutions in $C([0,T];L^2_w)$ with total kinetic energy e. In particular, if e is non-increasing, such solutions are dissipative [De Lellis and Székelyhidi '10].

Such non-uniqueness initial data for dissipative solutions are called wild initial data

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The set of wild initial data $v_0 \in L^\infty$ is dense in L^2 [Székelyhidi and Wiedemann '11].

Moreover, it includes the vortex sheet [Szekelyhidi '11].

Towards a regularity threshold

Definition

Given a divergence free vector field $v_0 \in C^{0,\theta_0}(\mathbb{T}^3)$, we say that v_0 is a wild initial datum in $C^{0,\theta}$ if there exist infinitely many admissible weak solutions v of (E) satisfying

$$|v(t,x) - v(t,y)| \leq C|x-y|^{\theta}, \quad \forall\, x, y \in \mathbb{T}^3,\ t \in [0,T].$$

Non-uniqueness up to $C^{1/3-\epsilon}$ for admissible weak solutions

Theorem (D. '14)

For every $\epsilon > 0$, there exist vector fields in $C^{0,1/10-\epsilon}$ which are wild initial data in $C^{0,1/16-\epsilon}$.
Moreover, they are infinitely many.

Theorem (D., Székelyhidi '16)

Let $\theta < \frac{1}{5}$. Then, there exist vector fields $v_0 \in C^{0,\theta}(\mathbb{T}^3)$ which are wild initial data in $C^{0,\theta}$.
Moreover, the set of such initial data is dense in $L^2(\mathbb{T}^3)$.

Theorem (D., Runa, Székelyhidi, In preparation)

Let $\theta < \frac{1}{3}$. Then, there exist vector fields $v_0 \in C^{0,\theta}(\mathbb{T}^3)$ which are wild initial data in $C^{0,\theta}$.
Moreover, the set of such initial data is dense in $L^2(\mathbb{T}^3)$.

1/3-scheme (case of no prescribed initial data)

Let $e \in C^\infty([0,T];\mathbb{R}^+)$. A strong subsolution of (E) w.r.t. e is $(v_q, p_q, \mathring{R}_q) \in C^\infty(\mathbb{T}^3 \times [0,T]; \mathbb{R}^3 \times \mathbb{R} \times S_0^{3\times 3})$ satisfying

$$(ER) \quad \begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div}\mathring{R}_q \\ \operatorname{div} v_q = 0 \end{cases}$$

$$\begin{aligned} &\|v_q - v_{q-1}\|_{C^0} \le \delta_q^{1/2} \\ &\|v_q\|_{C^1} \le \delta_q^{1/2}\lambda_q \\ &e(t) - \int_{\mathbb{T}^3} |v_q(t)|^2 \sim \delta_{q+1}, \quad \forall\, t \in [0,T] \\ &\|\mathring{R}_q\|_{C^0} \sim \delta_{q+1}\lambda_q^{-3\alpha}, \end{aligned}$$

for some η small geometric constant, $\lambda_q = a^{b^q}$, $\delta_q = \lambda_q^{-2\theta}$, $a >> 1$, $1 < b < 1+\epsilon$, $q \in \mathbb{N}$, $0 < \theta < 1/3$.

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for some η small geometric constant, $\lambda_q = a^{b^q}$, $\delta_q = \lambda_q^{-2\theta}$, $a >> 1$, $1 < b < 1 + \epsilon$, $q \in \mathbb{N}$, $0 < \theta < 1/3$.

Remark: If $q \to \infty$, then $(v_q, p_q, \mathring{R}_q)$ tends to a $C^{0,\theta}$-solution of the Euler equations with kinetic energy e.

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Hence, if we want to use a convex integration scheme leading to solutions with the same initial datum, we have to start from a concept of subsolution (adapted subsolution) that at time $t = 0$ is already a solution with energy $e(0)$ and then apply perturbations that at time $t = 0$ must all be null. This will answer the first point above.

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In order to show that such adapted subsolutions exist and are infinitely many, we perform another convex integration scheme starting from classical (strong) subsolutions adding perturbations which are each nonzero in smaller and smaller neighborhoods of $t = 0$.

From subsolutions to adapted subsolutions

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$$v_{q+1} \sim (1 - \psi_{q+1})v_q + \psi_{q+1}(\bar{v}_q + w_{q+1})$$

with $\psi_{q+1} \in C_c^\infty([0, T]; [0, 1])$ cut-off in time,

$$\psi_{q+1} = \begin{cases} 1 & \text{on } [0, 2^{-q}T] \\ 0 & \text{on } [2^{-(q-1)}T, T]. \end{cases} \tag{1}$$

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Since $\operatorname{supp} \psi_{q+2} \subset \{\psi_{q+1} = 1\}$, for the next step I can start with the uniform estimates of the $1/3$-scheme for dissipative solutions.
One has to show that on the remaining regions (not further modified) where $\psi_{q+1} \in (0, 1)$ one has a control on the norms growth, in particular that appearance of derivatives of the cut-off functions in the estimates for the Reynolds stress does not cause any problem.

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In the end, the sequence converges to $(v, p, \mathring{R})$ adapted subsolution defined by the following properties

Adapted subsolutions

Define $\rho = e - \int |v|^2$.

An adapted subsolution is $(v, p, \mathring{R}) \in C^\infty((0,T]) \cap C^0([0,T])$ solving (ER) on $(0,T]$ with

$$\int_{\mathbb{T}^3} |v_0|^2 = e(0), \qquad \mathring{R}(0) \equiv 0$$

and the following (non-uniform) estimates

$$\|\mathring{R}\|_0 \leq \rho^{1+\epsilon}$$
$$\|v\|_1 \leq \rho^{-1-\varepsilon}$$
$$|\partial_t \rho| \leq \rho^{-\epsilon}$$

From adapted subsolutions to solutions

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Idea: localize the estimates (with the aid of cut-off functions in time) to regions where the energy gap ρ is bounded from below $\Rightarrow$ uniform bound from above for the C^1-norms as in the $1/3$-scheme.

Density of wild initial data

Let $v_0 \in L^2$. Let $\bar{v} \in C^\infty$ s.t. $||v_0 - \bar{v}||_2 \leq \epsilon$ and let $(v, p, \mathring{R})$ a mollification of a solution of the Navier-Stokes equations with initial datum $\bar{v}$.

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In order to reduce to a subsolution with such a bound, we needed to introduce (in collaboration with Székelyhidi) the class of Mikado flows, which allow to "absorb" the error given by any positive definite matrix

$$R = \frac{1}{3}\Big(e(t) - \int |v|^2\Big)\mathrm{Id} - \mathring{R}$$

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Open problems

- What about well-posedness between $C^{0,1/3}$ and C^1?
- Are there selection criteria other than admissibility which allow to regain uniqueness?

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