Lack of uniqueness for the multi-dimensional compressible Euler equations

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Outline

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Compressible Euler equations

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho\, \mathbf{v}) &= 0 \\
\partial_t (\varrho\, \mathbf{v}) + \operatorname{div}(\varrho\, \mathbf{v} \otimes \mathbf{v}) + \nabla p &= 0 \\
\partial_t \Big(\tfrac{1}{2} \varrho |\mathbf{v}|^2 + \varrho\, e(\varrho, p) \Big) + \operatorname{div} \Big[\Big(\tfrac{1}{2} \varrho |\mathbf{v}|^2 + \varrho\, e(\varrho, p) + p \Big) \mathbf{v} \Big] &= 0
\end{aligned}
$$

Unknowns:

- density $\varrho : [0, T_{\max}) \times \Omega \to \mathbb{R}^+$
- velocity $\mathbf{v} : [0, T_{\max}) \times \Omega \to \mathbb{R}^n$
- pressure $p : [0, T_{\max}) \times \Omega \to \mathbb{R}^+$

Variables:

- time $t \in [0, T_{\max})$
- spatial variable $\mathbf{x} = (x_1, \dots, x_n) \in \Omega$
  $\Omega \subset \mathbb{R}^n$ bounded, $n = 2, 3$

The **internal energy** $e = e(\varrho, p)$ is a given function.

Example (Ideal gas)

$e(\varrho, p) = \frac{1}{\gamma - 1} \frac{p}{\varrho}$,
where $\gamma > 1$
(adiabatic exponent)

Initial boundary value problem

Initial condition:

$$\varrho(0,\cdot) = \varrho_0 \quad , \quad \mathbf{v}(0,\cdot) = \mathbf{v}_0 \quad , \quad p(0,\cdot) = p_0$$

Impermeability boundary condition:

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Conservation Laws

The Euler equations are a system of conservation laws:

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{F}(\mathbf{V}) = 0$$

with

$$\mathbf{V} = \begin{pmatrix} \varrho \\ \varrho\, \mathbf{v} \\ \frac{1}{2}\varrho|\mathbf{v}|^2 + \varrho\, e(\varrho, p) \end{pmatrix} \quad , \quad \mathbf{F} = \begin{pmatrix} \varrho\, \mathbf{v}^{\mathsf{T}} \\ \varrho\, \mathbf{v} \otimes \mathbf{v} + p\, \mathbb{I}_n \\ \big(\frac{1}{2}\varrho|\mathbf{v}|^2 + \varrho\, e(\varrho, p) + p\big)\mathbf{v}^{\mathsf{T}} \end{pmatrix}.$$

- We have to consider weak solutions.
- We need an admissibility criterion to select physically relevant solutions.

Weak solutions

Consider

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{F}(\mathbf{V}) = 0$$

with $\mathbf{V} : [0, T_{\max}) \times \Omega \to \mathbb{R}^m$, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^{m \times n}$.

Claim: A classical solution fulfills the integral identity

$$\int_0^{T_{\max}} \int_\Omega \mathbf{V} \cdot \partial_t \varphi + \mathbf{F}(\mathbf{V}) : \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_\Omega \mathbf{V}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}\mathbf{x} = 0$$

for all test functions $\varphi \in C_c^\infty([0, T_{\max}) \times \mathbb{R}^n; \mathbb{R}^m)$.

Weak solutions

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for all test functions $\varphi \in C_c^\infty([0, T_{\max}) \times \mathbb{R}^n; \mathbb{R}^m)$.

Proof: Multiply the PDE with φ, integrate and apply integration by parts. The support of φ is determined by the boundary condition!

Weak solutions

Consider

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{F}(\mathbf{V}) = 0$$

with $\mathbf{V} : [0, T_{\text{max}}) \times \Omega \to \mathbb{R}^m$, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^{m \times n}$.

Definition: Weak solution

A weak solution is a function $\mathbf{V} \in L^\infty([0, T_{\text{max}}) \times \Omega; \mathbb{R}^m)$ such that

$$\int_0^{T_{\text{max}}} \int_\Omega \mathbf{V} \cdot \partial_t \varphi + \mathbf{F}(\mathbf{V}) : \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_\Omega \mathbf{V}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}\mathbf{x} = 0$$

is fulfilled for all test functions $\varphi \in C_c^\infty([0, T_{\text{max}}) \times \mathbb{R}^n; \mathbb{R}^m)$.

Weak solutions to the Euler equations, 1

A weak solution is a triple of functions $(\varrho, \mathbf{v}, p) \in L^\infty([0, T_{\max}) \times \Omega; \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^+)$ such that

$$\int_0^{T_{\max}} \int_\Omega \left(\varrho\, \partial_t \psi + \varrho\, \mathbf{v} \cdot \nabla \psi\right) \, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_\Omega \varrho_0\, \psi(0, \cdot)\, \mathrm{d}\mathbf{x} \;=\; 0$$

$$\int_0^{T_{\max}} \int_\Omega \left(\varrho\, \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \varrho\, \mathbf{v} \otimes \mathbf{v} : \nabla \boldsymbol{\varphi} + p\, \operatorname{div} \boldsymbol{\varphi}\right) \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_\Omega \varrho_0\, \mathbf{v}_0 \cdot \boldsymbol{\varphi}(0, \cdot)\, \mathrm{d}\mathbf{x} \;=\; 0$$

for all test functions $(\psi, \boldsymbol{\varphi}) \in C_c^\infty([0, T_{\max}) \times \mathbb{R}^n, \mathbb{R} \times \mathbb{R}^n)$ with $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$,

Weak solutions to the Euler equations, 2

A weak solution is a triple of functions $(\varrho, \mathbf{v}, p) \in L^\infty([0, T_{\max}) \times \Omega; \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^+)$ such that

$$\int_0^{T_{\max}} \int_\Omega \Bigg(\Big(\tfrac{1}{2}\varrho|\mathbf{v}|^2 + \varrho\, e(\varrho, p) \Big)\, \partial_t \phi + \Big(\tfrac{1}{2}\varrho|\mathbf{v}|^2 + \varrho\, e(\varrho, p) + p \Big) \mathbf{v} \cdot \nabla \phi \Bigg)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_\Omega \Big(\tfrac{1}{2}\varrho_0|\mathbf{v}_0|^2 + \varrho_0\, e(\varrho_0, p_0) \Big) \phi(0, \cdot)\, \mathrm{d}\mathbf{x} \;=\; 0$$

for all test functions $\phi \in C_c^\infty([0, T_{\max}) \times \mathbb{R}^n, \mathbb{R})$.

Admissibility criterion

Consider

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{F}(\mathbf{V}) = 0$$

with $\mathbf{V} : [0, T_{\max}) \times \Omega \to \mathbb{R}^m$, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^{m \times n}$.

Definition: Entropy - entropy flux - pair

A pair of functions $(\eta, \psi) : \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^n$, $\mathbf{V} \mapsto (\eta(\mathbf{V}), \psi(\mathbf{V}))$ is called entropy - entropy flux - pair if

- η is a convex function and
- $\partial_{V_i} \psi_j = \sum_{k=1}^m \partial_{V_k} \eta \cdot \partial_{V_i} F_{kj}$.

Claim: Classical solutions fulfill $\partial_t \eta(\mathbf{V}) + \operatorname{div} \psi(\mathbf{V}) = 0$.

Proof:

Admissibility criterion

Consider

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{F}(\mathbf{V}) = 0$$

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Claim: Classical solutions fulfill $\partial_t \eta(\mathbf{V}) + \operatorname{div} \psi(\mathbf{V}) = 0$.

Proof:

$$\begin{aligned}
\partial_t \eta(\mathbf{V}) + \partial_{x_j} \psi_j(\mathbf{V}) &= \partial_{V_k} \eta \cdot \partial_t V_k + \partial_{V_i} \psi_j \cdot \partial_{x_j} V_i \\
&= \partial_{V_k} \eta \cdot \partial_t V_k + \partial_{V_k} \eta \cdot \partial_{V_i} F_{kj} \cdot \partial_{x_j} V_i \\
&= \partial_{V_k} \eta \cdot \Big(\partial_t V_k + \partial_{x_j} F_{kj} \Big) = 0
\end{aligned}$$

Admissibility criterion

Consider

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{F}(\mathbf{V}) = 0$$

with $\mathbf{V} : [0, T_{\max}) \times \Omega \to \mathbb{R}^m$, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^{m \times n}$.

Definition: Entropy - entropy flux - pair

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- η is a convex function and
- $\partial_{V_i} \psi_j = \sum_{k=1}^{m} \partial_{V_k} \eta \cdot \partial_{V_i} F_{kj}$.

Definition: Admissible solution (or entropy solution)

A weak solution is called admissible (or entropy solution) if

$$\partial_t \eta(\mathbf{V}) + \operatorname{div} \psi(\mathbf{V}) \leq 0$$

holds in the weak sense for all entropy - entropy flux - pairs (η, ψ).

Entropy solutions to the Euler equations

For the Euler equations

$$\eta = -\varrho\, s(\varrho, p) \qquad \boldsymbol{\psi} = -\varrho\, s(\varrho, p)\, \mathbf{v}$$

is an entropy - entropy flux - pair.

Ideal gas

$s(\varrho, p) = \frac{1}{\gamma - 1} \log p - \frac{\gamma}{\gamma - 1} \log \varrho$

A weak solution is admissible if

$$\int_0^{T_{\max}} \int_\Omega \Big(\varrho\, s(\varrho, p)\, \partial_t \varphi + \varrho\, s(\varrho, p)\, \mathbf{v} \cdot \nabla \varphi \Big)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_\Omega \Big(\varrho_0\, s(\varrho_0, p_0) \Big) \varphi(0, \cdot)\, \mathrm{d}\mathbf{x} \;\leq\; 0$$

for all test functions $\varphi \in C_c^\infty([0, T_{\max}) \times \mathbb{R}^n, [0, \infty))$.

Isentropic Euler equations

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{v}) = 0$$
$$\partial_t(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\varrho) = 0$$

Unknowns:

- density $\varrho : [0, T_{\max}) \times \Omega \to \mathbb{R}^+$
- velocity $\mathbf{v} : [0, T_{\max}) \times \Omega \to \mathbb{R}^n$

Variables:

- time $t \in [0, T_{\max})$
- spatial variable $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$
  $\Omega \subset \mathbb{R}^n$ bounded, $n = 2, 3$

The **pressure** $p = p(\varrho)$ and the **pressure potential** $P = P(\varrho)$ are given functions.

Example (Polytropic pressure law)

$p(\varrho) = \varrho^\gamma$, $P(\varrho) = \frac{1}{\gamma - 1}\varrho^\gamma$, where $\gamma > 1$

Entropy (energy) inequality:

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{v}|^2 + P(\varrho) \right) + \operatorname{div}\left[\left(\frac{1}{2}\varrho|\mathbf{v}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{v} \right] \leq 0$$

Outline

Results for the isentropic Euler equations

Theorem

Consider the isentropic Euler equations with an arbitrary pressure function $p(\varrho)$. There exist initial data $(\varrho_0, \mathbf{v}_0)$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{v})$.

C. De Lellis and L. Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". In: Arch. Ration. Mech. Anal. 195.1 (2010), pp. 225–260

Theorem

Consider the isentropic Euler equations with an arbitrary pressure function $p(\varrho)$. For any given periodic initial density $\varrho_0 \in C^1$ there exist a periodic initial momentum $\mathbf{m}_0 \in L^\infty$ and a positive time $T_{\max}$ for which there are infinitely many space-periodic admissible weak solutions $(\varrho, \mathbf{m})$ on $[0, T_{\max}) \times \mathbb{R}^n$.

E. Chiodaroli. "A counterexample to well-posedness of entropy solutions to the compressible Euler system". In: J. Hyperbolic Differ. Equ. 11.3 (2014), pp. 493–519

Results for the full Euler equations

Theorem

For any given piecewise-constant initial density ϱ_0 and pressure p_0 there exists an initial velocity $\mathbf{v}_0 \in L^\infty$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{v}, p)$ on $[0, T_{\max}) \times \Omega$.

O. Kreml E. Feireisl C. Klingenberg and S. Markfelder. "On oscillatory solutions to the complete Euler system". In: submitted (2017). arXiv: 1710.10918

Theorem (the one we are going to prove)

For any given constant initial density ϱ_0 and pressure p_0 there exists an initial velocity $\mathbf{v}_0 \in L^\infty$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{v}, p)$ on $[0, T_{\max}) \times \Omega$.

Outline

Basic idea

De Lellis and Székelyhidi showed existence of infinitely many solutions $(\mathbf{v}, p)$ to the *incompressible* Euler equations

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla p &= 0, \end{aligned}$$

where one can achieve $p \equiv \text{const}$ and prescribe the kinetic energy $|\mathbf{v}(t, \mathbf{x})|^2 = \overline{e}(t, \mathbf{x})$ for a. e. $(t, \mathbf{x})$.

C. De Lellis and L. Székelyhidi Jr. "The Euler equations as a differential inclusion". In: *Ann. of Math. (2)* 170.3 (2009), pp. 1417–1436

C. De Lellis and L. Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". In: *Arch. Ration. Mech. Anal.* 195.1 (2010), pp. 225–260

Idea

For the *compressible* Euler equations set $\varrho \equiv \text{const}$, $\overline{e} \equiv \text{const}$ and use their result.

Proposition 1

Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) open and bounded, $0 < T < \infty$ and $r > 0$, $c > 0$ positive constants. Then there exists $\mathbf{m}_0, \mathbf{m}_T \in L^\infty(\Omega; \mathbb{R}^n)$ such that the problem

$$\begin{aligned} \operatorname{div} \mathbf{m} &= 0 \\ \partial_t \mathbf{m} + \operatorname{div} \left(\frac{\mathbf{m} \otimes \mathbf{m}}{r} - \frac{1}{n} \frac{|\mathbf{m}|^2}{r} \mathbb{I}_n \right) &= 0 \\ \mathbf{m}(0, \cdot) &= \mathbf{m}_0 \\ \mathbf{m}(T, \cdot) &= \mathbf{m}_T \end{aligned}$$

has infinitely many weak solutions that fulfill $\frac{|\mathbf{m}|^2}{r} = c$ for a. e. $(t, \mathbf{x}) \in [0, T] \times \Omega$.

Proposition 1

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ open and bounded, $0 < T < \infty$ and $r > 0$, $c > 0$ positive constants. Then there exists $\mathbf{m}_0, \mathbf{m}_T \in L^\infty(\Omega; \mathbb{R}^n)$ such that there are infinitely many

$$\mathbf{m} \in L^\infty((0,T) \times \Omega; \mathbb{R}^n) \cap C_{\text{weak}}([0,T]; L^2(\Omega; \mathbb{R}^n))$$

with

$$\int_0^T \int_\Omega \mathbf{m} \cdot \nabla\psi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = 0$$

$$\int_0^T \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \left(\frac{\mathbf{m} \otimes \mathbf{m}}{r} - \frac{1}{n} \frac{|\mathbf{m}|^2}{r} \mathbb{I}_n \right) : \nabla \boldsymbol{\varphi} \right] \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{m}_T \cdot \boldsymbol{\varphi}(T, \cdot) \, \mathrm{d}\mathbf{x} = 0$$

for all test functions $(\psi, \boldsymbol{\varphi}) \in C_c^\infty([0,T] \times \mathbb{R}^n, \mathbb{R} \times \mathbb{R}^n)$ and

$$\frac{|\mathbf{m}|^2}{r} = c \quad \text{for a. e. } (t, \mathbf{x}) \in [0,T] \times \Omega.$$

Proof of the theorem

Proposition 1*

Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) open and bounded, $0 < T < \infty$ and $r > 0$, $c > 0$ positive constants. Then there exists $\mathbf{m}_0, \mathbf{m}_T \in L^\infty(\Omega; \mathbb{R}^n)$ such that there are infinitely many

$$\mathbf{m} \in L^\infty((0,T) \times \Omega; \mathbb{R}^n) \cap C_{\mathrm{weak}}([0,T]; L^2(\Omega; \mathbb{R}^n))$$

with

$$\int_0^T \int_\Omega \mathbf{m} \cdot \nabla\psi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = 0$$

$$\int_0^T \int_\Omega \left[\mathbf{m} \cdot \partial_t \varphi + \left(\frac{\mathbf{m} \otimes \mathbf{m}}{r} - \frac{1}{n} \frac{|\mathbf{m}|^2}{r} \mathbb{I}_n \right) : \nabla\varphi \right] \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_\Omega \mathbf{m}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{m}_T \cdot \varphi(T, \cdot) \, \mathrm{d}\mathbf{x} = 0$$

for all test functions $(\psi, \varphi) \in C_c^\infty([0,T] \times \mathbb{R}^n, \mathbb{R} \times \mathbb{R}^n)$ and

$$\frac{|\mathbf{m}|^2}{r} = c \quad \text{for all } t \in [0,T] \text{ and a. e. } \mathbf{x} \in \Omega.$$

Outline

References

C. De Lellis and L. Székelyhidi Jr. "The Euler equations as a differential inclusion". In: *Ann. of Math. (2)* 170.3 (2009), pp. 1417–1436

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E. Feireisl. "Weak solutions to problems involving inviscid fluids". In: *Mathematical Fluid Dynamics, Present and Future*. Vol. 183. Springer Proceedings in Mathematics and Statistics. Tokyo: Springer-Verlag, 2016, pp. 377–399

Basic ideas of the convex integration method

$$\begin{aligned} \operatorname{div} \mathbf{m} &= 0 \\ \partial_t \mathbf{m} + \operatorname{div}\left(\frac{\mathbf{m} \otimes \mathbf{m}}{r} - \frac{1}{n} \frac{|\mathbf{m}|^2}{r} \mathbb{I}_n \right) &= 0 \end{aligned}$$

Basic ideas of the convex integration method

$$\operatorname{div} \mathbf{m} = 0$$
$$\partial_t \mathbf{m} + \operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{r} - \frac{1}{n}\frac{|\mathbf{m}|^2}{r}\mathbb{I}_n\right) = 0$$

1. Rewrite the system as a linear one with a non-linear constraint by introducing the new unknown $U \in \mathcal{S}_0^n$:

$$\operatorname{div} \mathbf{m} = 0,$$
$$\partial_t \mathbf{m} + \operatorname{div} U = 0,$$

with the non-linear constraint $(\mathbf{m}, U) \in Z$ where

$$Z := \Big\{(\mathbf{m}, U) : [0, T] \times \Omega \to \mathbb{R}^n \times \mathcal{S}_0^n \,\Big|\, \big(\mathbf{m}(t, \mathbf{x}), U(t, \mathbf{x})\big) \in K \text{ for almost all } (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^n\Big\},$$
$$K := \Big\{(\mathbf{m}, U) \in \mathbb{R}^n \times \mathcal{S}_0^n \,\Big|\, U = \frac{\mathbf{m}\otimes\mathbf{m}}{r} - \frac{c}{n}\,\mathbb{I}_n\Big\}.$$

Basic ideas of the convex integration method

2. Relax the constraint: $Z \mapsto \widehat{Z}$, with

$$\widehat{Z} := \Big\{(\mathbf{m}, U) : [0,T] \times \Omega \to \mathbb{R}^n \times \mathcal{S}_0^n \,\Big|\, \big(\mathbf{m}(t,\mathbf{x}), U(t,\mathbf{x})\big) \in (K^{\mathrm{co}})^\circ \text{ for almost all } (t,\mathbf{x}) \in [0,T] \times \mathbb{R}^n\Big\}.$$

Weak solutions to the linearized system are called **subsolutions** if they fulfill the relaxed constraint.

Basic ideas of the convex integration method

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3. Find a subsolution $(\overline{\mathbf{m}}, \overline{U})$.

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   Weak solutions to the linearized system are called **subsolutions** if they fulfill the relaxed constraint.

3. Find a subsolution $(\overline{\mathbf{m}}, \overline{U})$.

Constructive approach

4. Construct a sequence of subsolutions $(\mathbf{m}_k, U_k)_k$ (where $(\mathbf{m}_0, U_0) = (\overline{\mathbf{m}}, \overline{U})$) converging to a solution $(\mathbf{m}, U)$.

Basic ideas of the convex integration method

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Baire category approach

4. Prove - by using Baire category arguments - that if a subsolution exists then there are infinitely many solutions.

Geometric setup

Define

$$e : \mathbb{R}^n \times \mathcal{S}_0^n \to \mathbb{R}, \quad e(\mathbf{m}, U) = \lambda_{\max}\left(\frac{\mathbf{m} \otimes \mathbf{m}}{r} - U\right)$$

$$\mathcal{U} := \left\{(\mathbf{m}, U) \in \mathbb{R}^n \times \mathcal{S}_0^n \,\middle|\, e(\mathbf{m}, U) < \tfrac{c}{n}\right\}$$

$$K := \left\{(\mathbf{m}, U) \in \mathbb{R}^n \times \mathcal{S}_0^n \,\middle|\, \tfrac{\mathbf{m}\otimes\mathbf{m}}{r} - U = \tfrac{c}{n}\,\mathbb{I}_n\right\}$$

Properties:

- $\frac{|\mathbf{m}|^2}{r\,n} \leq e(\mathbf{m}, U)$ for all $(\mathbf{m}, U) \in \mathbb{R}^n \times \mathcal{S}_0^n$
- $\frac{|\mathbf{m}|^2}{r\,n} = e(\mathbf{m}, U) \Leftrightarrow U = \frac{\mathbf{m}\otimes\mathbf{m}}{r} - \frac{|\mathbf{m}|^2}{r\,n}\mathbb{I}_n$
- e is a convex function
- $|U|_\infty \leq (n-1)\, e(\mathbf{m}, U)$ for all $(\mathbf{m}, U) \in \mathbb{R}^n \times \mathcal{S}_0^n$
- $\mathcal{U} = (K^{\text{co}})^\circ$, where K^{co} denotes the convex hull of K.

Proposition 2

Assume there exists a smooth solution $(\overline{\mathbf{m}}, \overline{U})$ of the system

$$\begin{aligned} \operatorname{div} \overline{\mathbf{m}} &= 0 \\ \partial_t \overline{\mathbf{m}} + \operatorname{div} \overline{U} &= 0 \end{aligned}$$

with the following properties

$$\begin{aligned} &\overline{\mathbf{m}} \in C_{\text{weak}}([0, T]; L^2(\mathbb{R}^n; \mathbb{R}^n)) \\ &\operatorname{supp}\left(\overline{\mathbf{m}}(t, \cdot), \overline{U}(t, \cdot)\right) \subset \Omega \quad \text{for all } t \in (0, T) \\ &e(\overline{\mathbf{m}}, \overline{U}) < \tfrac{c}{n} \quad \text{for all } (t, \mathbf{x}) \in (0, T) \times \Omega. \end{aligned}$$

Then there exist infinitely many solutions $\mathbf{m}$ as in Proposition 1 such that

$$\begin{aligned} \mathbf{m}(t, \cdot) = \overline{\mathbf{m}}(t, \cdot) &\quad \text{for } t = 0, T \\ \frac{|\mathbf{m}(t, \mathbf{x})|^2}{r} = c &\quad \text{for almost every } (t, \mathbf{x}) \in (0, T) \times \Omega \end{aligned}$$

Proof of Proposition 1

Proof of Proposition 2

Define the set

$$X_0 := \Big\{ \mathbf{m} \in C^\infty((0,T)\times\mathbb{R}^n;\mathbb{R}^n) \cap C_{\text{weak}}([0,T];L^2(\mathbb{R}^n;\mathbb{R}^n)) \,\Big|\, \text{conditions C1,C2,C3 hold} \Big\}.$$

C1 $\operatorname{div}\mathbf{m} = 0$

C2 $\mathbf{m}(t,\cdot) = \overline{\mathbf{m}}(t,\cdot)$ for $t = 0, T$
$\operatorname{supp}\mathbf{m}(t,\cdot) \subset \Omega$ for all $t \in (0,T)$

C3 there exists $U \in C^\infty((0,T)\times\mathbb{R}^n;\mathcal{S}_0^n)$ with

- $\operatorname{supp} U(t,\cdot) \subset \Omega$ for all $t \in (0,T)$
- $e(\mathbf{m}(t,\mathbf{x}), U(t,\mathbf{x})) < \frac{c}{n}$ for all $(t,\mathbf{x}) \in (0,T)\times\Omega$
- $\partial_t \mathbf{m} + \operatorname{div} U = 0$ in $(0,T)\times\Omega$.

Proof of Proposition 2

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$$X_0 := \Big\{ \mathbf{m} \in C^\infty((0,T)\times\mathbb{R}^n;\mathbb{R}^n) \cap C_{\mathrm{weak}}([0,T];L^2(\mathbb{R}^n;\mathbb{R}^n)) \,\Big|\, \text{conditions C1,C2,C3 hold} \Big\}.$$

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C3 there exists $U \in C^\infty((0,T)\times\mathbb{R}^n;\mathcal{S}_0^n)$ with

- $\operatorname{supp} U(t,\cdot) \subset \Omega$ for all $t \in (0,T)$
- $(\mathbf{m}(t,\mathbf{x}), U(t,\mathbf{x})) \in \mathcal{U}$ for all $(t,\mathbf{x}) \in (0,T)\times\Omega$
- $\partial_t \mathbf{m} + \operatorname{div} U = 0$ in $(0,T)\times\Omega$.

Proof of Proposition 2

Let $\mathbf{m} \in X_0$. Then

$$\begin{aligned} \|\mathbf{m}(t,\cdot)\|_{L^2}^2 &= \int_\Omega |\mathbf{m}(t,\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \\ &\leq n\, r \int_\Omega e(\mathbf{m}(t,\mathbf{x}), U(t,\mathbf{x})) \, \mathrm{d}\mathbf{x} \\ &< c\, r\, |\Omega| && \text{for all } t \in (0,T) \\ \|\mathbf{m}(t,\cdot)\|_{L^2}^2 &= \|\overline{\mathbf{m}}(t,\cdot)\|_{L^2}^2 && \text{for } t = 0, T. \end{aligned}$$

▷ $\mathbf{m} : [0,T] \to L^2$ takes values in a bounded subset $B \subset L^2$.

Proof of Proposition 2

Let $\mathbf{m} \in X_0$. Then

- $\mathbf{m} : [0, T] \to L^2$ takes values in a bounded subset $B \subset L^2$.
- W.l.o.g. assume that B is closed in the weak topology of L^2 (otherwise consider the weak closure of B).
- The weak topology on B is metrizable (denote this metric by d_B) and (B, d_B) is a compact metric space (Alaoglu's theorem).
- Hence (B, d_B) is a complete metric space.
- Define the metric d on $C([0, T]; (B, d_B))$ by

$$d(\mathbf{m}_1, \mathbf{m}_2) := \max_{t \in [0,T]} d_B(\mathbf{m}_1(t, \cdot), \mathbf{m}_2(t, \cdot)).$$

- Then $(C([0, T]; (B, d_B)), d)$ is a complete metric space, too.
- Let X be the closure of X_0 w. r. t. the metric d.
- Then (X, d) is a complete metric space.

Proof of Proposition 2

Lemma

Let $\mathbf{m} \in X$ *such that* $\frac{|\mathbf{m}|^2}{r} = c$ *for a. e.* $(t, \mathbf{x}) \in (0, T) \times \Omega$ *then* $\mathbf{m}$ *is a solution as in Proposition 2.*

Proof:

Baire category theory

- Let $(M, \mathcal{T})$ be a topological space. A subset $A \subset M$ is called
  - *nowhere dense* if the interior of the closure of A is empty:
  $$(\overline{A})^\circ = \varnothing,$$
  - *meager* (or *of first category*) if A is the countable union of nowhere dense sets,
  - *residual* if the complement of A is meager.
- Baire category theorem: If (M, d) is a complete metric space, then every residual subset of M is dense.
- Let $(M_1, \mathcal{T})$ a topological and (M_2, d) a metric space. A function $f : M_1 \to M_2$ is called Baire-1-function if it is the pointwise limit of a sequence of continuous functions.
- Let $(M_1, \mathcal{T})$ a topological and $(M_2, \|\cdot\|)$ a normed space and consider a Baire-1-function $f : M_1 \to M_2$. Then the set $C \subset M_1$ of the points in which f is continuous is residual in M_1.

Proof of Proposition 2

Plan of the proof:

- Because of the lemma, each $\mathbf{m} \in Y$ is a solution, where

$$Y := \left\{ \mathbf{m} \in X \,\middle|\, \frac{|\mathbf{m}|^2}{r} = c \text{ for a.e. } (t, \mathbf{x}) \in (0, T) \times \Omega \right\}.$$

- Show that the identity map $I : (X, d) \to \big(L^2, \|\cdot\|_{L^2}\big)$, $\mathbf{m} \mapsto \mathbf{m}$ is a Baire-1-function.
- The set $C := \{\mathbf{m} \in X \,|\, I \text{ is continuous in } \mathbf{m}\}$ is residual in X.
- Show that $C \subset Y$.
- Since (X, d) is a complete metric space, C is dense (Baire category theorem). Hence Y is dense.
- Show that X is infinite. Then Y is infinite, too.

Proof of Proposition 2

Claim: $I : (X, d) \to \left(L^2, \|\cdot\|_{L^2}\right)$, $\mathbf{m} \mapsto \mathbf{m}$ is a Baire-1-function.

Proof of Proposition 2

Claim: $I : (X, d) \to \left(L^2, \|\cdot\|_{L^2}\right)$, $\mathbf{m} \mapsto \mathbf{m}$ is a Baire-1-function.

Let $I_\delta : (X, d) \to \left(L^2, \|\cdot\|_{L^2}\right)$ defined by $\mathbf{m} \mapsto \Phi_\delta * \mathbf{m}$ with a space-time mollifier Φ_δ. One can show that

- $\mathbf{m}_k \overset{d}{\to} \mathbf{m}$ implies $\Phi_\delta * \mathbf{m}_k \to \Phi_\delta * \mathbf{m}$ in L^2,
- $\Phi_\delta * \mathbf{m} \to \mathbf{m}$ in L^2 for $\delta \to 0$.

Hence the functions I_δ are continuous and converge pointwise to I as $\delta \to 0$.

Proof of Proposition 2

Claim: $C \subset Y$.

$$Y := \left\{ \mathbf{m} \in X \,\middle|\, \frac{|\mathbf{m}|^2}{r} = c \text{ for a.e. } (t, \mathbf{x}) \in (0, T) \times \Omega \right\}$$

$$C := \left\{ \mathbf{m} \in X \,\middle|\, I \text{ is continuous in } \mathbf{m} \right\}$$

Proof of Proposition 2

Claim: $C \subset Y$.

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$$C := \left\{ \mathbf{m} \in X \;\middle|\; I \text{ is continuous in } \mathbf{m} \right\}$$

Oscillatory Lemma

For all compact sets $\Gamma \subset (0, T) \times \Omega$ there exists a constant $\beta > 0$ with the following property. For any given $\mathbf{m} \in X_0$ there exists a sequence $(\mathbf{m}_k)_{k \in \mathbb{N}} \subset X_0$ such that

$$\mathbf{m}_k \overset{d}{\to} \mathbf{m}$$

$$\liminf_{k \to \infty} \|\mathbf{m}_k\|^2_{L^2(\Gamma)} \geq \|\mathbf{m}\|^2_{L^2(\Gamma)} + \beta \left(c\, r\, |\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)} \right)^2.$$

Proof of Proposition 2

Claim: X is infinite.

Proof of Proposition 2

Claim: X is infinite.

- Since $\overline{\mathbf{m}} \in X_0$, $X_0 \neq \varnothing$.
- From the oscillatory lemma we can deduce that X_0 is infinite.
- Hence X is infinite.

Outline

Oscillatory lemma

Lemma

For all compact sets $\Gamma \subset (0, T) \times \Omega$ *there exists a constant* $\beta > 0$ *with the following property. For any given* $\mathbf{m} \in X_0$ *there exists a sequence* $(\mathbf{m}_k)_{k\in\mathbb{N}} \subset X_0$ *such that*

$$\mathbf{m}_k \overset{d}{\to} \mathbf{m}$$

$$\liminf_{k\to\infty} \|\mathbf{m}_k\|^2_{L^2(\Gamma)} \geq \|\mathbf{m}\|^2_{L^2(\Gamma)} + \beta \left(c\, r\, |\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)} \right)^2.$$

Proof of the Oscillatory lemma, Step 1

Define

$$n^* := \dim(\mathbb{R}^n \times \mathcal{S}_0^n) + 1 = n + \sum_{i=1}^{n} i - 1 + 1 = \frac{n(n+3)}{2}.$$

Fix an arbitrary point $(t_0, \mathbf{x}_0) \in \Gamma$. For convenience we define

$$(\mathbf{m}^\star, U^\star) := \big(\mathbf{m}(t_0, \mathbf{x}_0), U(t_0, \mathbf{x}_0)\big).$$

By assumption it holds that $(\mathbf{m}^\star, U^\star) \in \mathcal{U}$.

Proof of the Oscillatory lemma, Step 1

Claim: $\exists$ a segment $\sigma_{t_0,\mathbf{x}_0} = [-p, p] \subset \mathbb{R}^n \times \mathcal{S}^n_0$ such that:

1. $\exists\ \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $|\mathbf{a}| = |\mathbf{b}| = \sqrt{r\,c}$ and $\mathbf{a} \neq \pm\mathbf{b}$, and $\lambda > 0$ such that
$$p = \lambda\left[\left(\mathbf{a}, \tfrac{\mathbf{a}\otimes\mathbf{a}}{r}\right) - \left(\mathbf{b}, \tfrac{\mathbf{b}\otimes\mathbf{b}}{r}\right)\right].$$
2. $(\mathbf{m}^\star, U^\star) + \sigma_{t_0,\mathbf{x}_0} \subset \mathcal{U}$.
3. $\forall\ \varepsilon > 0\ \exists$ a pair $(\mathbf{m}_{t_0,\mathbf{x}_0}, U_{t_0,\mathbf{x}_0}) \in C^\infty_c\big((-1,1) \times B_1(0)\big)$ s. t.
   - $\operatorname{div} \mathbf{m}_{t_0,\mathbf{x}_0} = 0 \qquad \partial_t \mathbf{m}_{t_0,\mathbf{x}_0} + \operatorname{div} U_{t_0,\mathbf{x}_0} = 0$
   - $\operatorname{dist}\big((\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x}), U_{t_0,\mathbf{x}_0}(t,\mathbf{x})), \sigma_{t_0,\mathbf{x}_0}\big) < \varepsilon$ for all $(t,\mathbf{x}) \in (-1,1) \times B_1(0)$
   - $\displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \geq c_1\left(r\,c - \left|\mathbf{m}(t_0,\mathbf{x}_0)\right|^2\right)$ for a suitable constant $c_1 > 0$
   - $\displaystyle\int_{\mathbb{R}^n} \mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$

Proof of the Oscillatory lemma, Step 1

Claim: $\exists$ a segment $\sigma_{t_0,\mathbf{x}_0} = [-p, p] \subset \mathbb{R}^n \times \mathcal{S}_0^n$ such that:

1. $\exists$ $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $|\mathbf{a}| = |\mathbf{b}| = \sqrt{r\,c}$ and $\mathbf{a} \neq \pm\mathbf{b}$, and $\lambda > 0$ such that

$$p = \lambda \left[\left(\mathbf{a}, \tfrac{\mathbf{a}\otimes\mathbf{a}}{r}\right) - \left(\mathbf{b}, \tfrac{\mathbf{b}\otimes\mathbf{b}}{r}\right) \right].$$

Proof:

- ▷ We have that $(\mathbf{m}^\star, U^\star) \in \mathcal{U} = (K^{\mathrm{co}})^\circ$.
- ▷ $\exists$ finitely many $(\mathbf{m}_i, U_i) \in K$ such that $(\mathbf{m}^\star, U^\star)$ lies in the interior of the convex polytope spanned by the $(\mathbf{m}_i, U_i)$.
- ▷ Since $(\mathbf{m}^\star, U^\star)$ lies in the interior, it is possible to slightly change the $(\mathbf{m}_i, U_i)$ to obtain $\mathbf{m}_i \neq \pm\mathbf{m}_j$ for all $i \neq j$.
- ▷ By Caratheodory's theorem, there are at most n^* points among the $(\mathbf{m}_i, U_i)$ and $\alpha_i \geq 0$ such that

$$(\mathbf{m}^\star, U^\star) = \sum_{i=1}^{n^*} \alpha_i\,(\mathbf{m}_i, U_i), \qquad \sum_{i=1}^{n^*} \alpha_i = 1.$$

Proof of the Oscillatory lemma, Step 1

Claim: $\exists$ a segment $\sigma_{t_0,\mathbf{x}_0} = [-p, p] \subset \mathbb{R}^n \times \mathcal{S}^n_0$ such that:

1. $\exists\ \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $|\mathbf{a}| = |\mathbf{b}| = \sqrt{r\,c}$ and $\mathbf{a} \neq \pm\mathbf{b}$, and $\lambda > 0$ such that
$$p = \lambda\left[\left(\mathbf{a}, \tfrac{\mathbf{a}\otimes\mathbf{a}}{r}\right) - \left(\mathbf{b}, \tfrac{\mathbf{b}\otimes\mathbf{b}}{r}\right)\right].$$

Proof:

- By Caratheodory's theorem, there are at most n^* points among the $(\mathbf{m}_i, U_i)$ and $\alpha_i \geq 0$ such that
$$(\mathbf{m}^\star, U^\star) = \sum_{i=1}^{n^*} \alpha_i\,(\mathbf{m}_i, U_i), \qquad \sum_{i=1}^{n^*} \alpha_i = 1.$$
- Since $(\mathbf{m}^\star, U^\star) \notin K$, there are at least two indices i with $\alpha_i > 0$. W.l.o.g. the coefficients are ordered such that $\alpha_1 = \max_i \alpha_i$.
- Let j be such that $\alpha_j\,|\mathbf{m}_j - \mathbf{m}_1| = \max_i \alpha_i\,|\mathbf{m}_i - \mathbf{m}_1|$.

Proof of the Oscillatory lemma, Step 1

Claim: $\exists$ a segment $\sigma_{t_0,\mathbf{x}_0} = [-p, p] \subset \mathbb{R}^n \times \mathcal{S}_0^n$ such that:

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$$p = \lambda \left[\left(\mathbf{a}, \tfrac{\mathbf{a}\otimes\mathbf{a}}{r}\right) - \left(\mathbf{b}, \tfrac{\mathbf{b}\otimes\mathbf{b}}{r}\right) \right].$$

Proof:

- Let j be such that $\alpha_j\,|\mathbf{m}_j - \mathbf{m}_1| = \max_i \alpha_i\,|\mathbf{m}_i - \mathbf{m}_1|$.
- Set $\mathbf{a} = \mathbf{m}_j$, $\mathbf{b} = \mathbf{m}_1$. Note that $j \neq 1$ and hence $\mathbf{a} \neq \pm\mathbf{b}$.
- We obtain that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{r\,c}$ because $(\mathbf{m}_i, U_i) \in K$ and therefore $|\mathbf{m}_i|^2 = \operatorname{tr}(\mathbf{m}_i \otimes \mathbf{m}_i) = r\ \operatorname{tr}(\frac{c}{n}\,\mathbb{I}_n + U_i) = r\,c$ (for all $i \in \{1, \ldots, n^*\}$).
- We set $\lambda = \frac{1}{2}\,\alpha_j$ and $p = \lambda \left[\left(\mathbf{a}, \frac{\mathbf{a}\otimes\mathbf{a}}{r}\right) - \left(\mathbf{b}, \frac{\mathbf{b}\otimes\mathbf{b}}{r}\right) \right]$. Then $p \in \mathbb{R}^n \times \mathcal{S}_0^n$ since $\frac{\lambda}{r}(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b})$ is symmetric and $\operatorname{tr}\left(\lambda(\frac{\mathbf{a}\otimes\mathbf{a}}{r} - \frac{\mathbf{b}\otimes\mathbf{b}}{r})\right) = \frac{\lambda}{r}\left(|\mathbf{a}|^2 - |\mathbf{b}|^2\right) = 0$.

Proof of the Oscillatory lemma, Step 1

2. $(\mathbf{m}^\star, U^\star) + \sigma_{t_0,\mathbf{x}_0} \subset \mathcal{U}$.

Proof:

Proof of the Oscillatory lemma, Step 1

Additionally the following estimates hold:

Proof of the Oscillatory lemma, Step 1

Additionally the following estimates hold:
Since $\alpha_j\,|\mathbf{m}_j - \mathbf{m}_1| = \max_i \alpha_i\,|\mathbf{m}_i - \mathbf{m}_1|$, we get that

$$
\begin{aligned}
|\mathbf{m}^\star - \mathbf{m}_1| &= \left| \sum_{i=1}^{n^*} \alpha_i\,\mathbf{m}_i - \sum_{i=1}^{n^*} \alpha_i\,\mathbf{m}_1 \right| \\
&= \left| \sum_{i=1}^{n^*} \alpha_i\,(\mathbf{m}_i - \mathbf{m}_1) \right| \\
&\leq \sum_{i=1}^{n^*} \alpha_i\,|\mathbf{m}_i - \mathbf{m}_1| \\
&\leq n^*\,\alpha_j\,|\mathbf{m}_j - \mathbf{m}_1|.
\end{aligned}
$$

Proof of the Oscillatory lemma, Step 1

Additionally the following estimates hold:
Since $\alpha_j \, |\mathbf{m}_j - \mathbf{m}_1| = \max_i \alpha_i \, |\mathbf{m}_i - \mathbf{m}_1|$, we get that

$$|\mathbf{m}^\star - \mathbf{m}_1| \leq n^* \, \alpha_j \, |\mathbf{m}_j - \mathbf{m}_1|.$$

Proof of the Oscillatory lemma, Step 1

Additionally the following estimates hold:
Since $\alpha_j\,|\mathbf{m}_j - \mathbf{m}_1| = \max_i \alpha_i\,|\mathbf{m}_i - \mathbf{m}_1|$, we get that

$$|\mathbf{m}^\star - \mathbf{m}_1| \leq n^*\,\alpha_j\,|\mathbf{m}_j - \mathbf{m}_1|.$$

Hence:

$$\begin{aligned}
\lambda\,|\mathbf{a} - \mathbf{b}| &= \frac{1}{2}\,\alpha_j\,|\mathbf{m}_j - \mathbf{m}_1| \geq \frac{1}{2}\;\frac{1}{n^*}\,|\mathbf{m}^\star - \mathbf{m}_1| \\
&\geq \frac{1}{2n^*}\,\big(|\mathbf{m}_1| - |\mathbf{m}^\star|\big) > \frac{1}{2n^*}\,\big(\sqrt{r\,c} - |\mathbf{m}^\star|\big)\,\frac{\sqrt{r\,c} + |\mathbf{m}^\star|}{2\,\sqrt{r\,c}} \\
&= \frac{1}{4n^*\,\sqrt{r\,c}}\,\big(r\,c - |\mathbf{m}^\star|^2\big),
\end{aligned}$$

where we used that $|\mathbf{m}^\star|^2 \leq r\,n\,e(\mathbf{m}^\star, U^\star) < r\,c$.

Proof of the Oscillatory lemma, Step 1

3. $\forall\, \varepsilon > 0\ \exists$ a pair $(\mathbf{m}_{t_0,\mathbf{x}_0}, U_{t_0,\mathbf{x}_0}) \in C_c^\infty\big((-1,1)\times B_1(0)\big)$ s. t.

   - $\operatorname{div} \mathbf{m}_{t_0,\mathbf{x}_0} = 0 \qquad \partial_t \mathbf{m}_{t_0,\mathbf{x}_0} + \operatorname{div} U_{t_0,\mathbf{x}_0} = 0$
   - $\operatorname{dist}\big((\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x}), U_{t_0,\mathbf{x}_0}(t,\mathbf{x})), \sigma_{t_0,\mathbf{x}_0}\big) < \varepsilon$
     for all $(t,\mathbf{x}) \in (-1,1)\times B_1(0)$
   - $\displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}^n} \big|\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x})\big|\ \mathrm{d}\mathbf{x}\ \mathrm{d}t \geq c_1\left(r\,c - \big|\mathbf{m}(t_0,\mathbf{x}_0)\big|^2\right)$
     for a suitable constant $c_1 > 0$
   - $\displaystyle\int_{\mathbb{R}^n} \mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x})\ \mathrm{d}\mathbf{x} = 0.$

Proof of the Oscillatory lemma, Step 1

Lemma (De Lellis, Székelyhidi)

There exist linear differential operators of order 3

$$A : C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}) \to C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^n)$$
$$B : C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}) \to C_c^\infty(\mathbb{R}^{n+1};\mathcal{S}_0^n)$$

s. t. for all $\Phi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R})$

$$\operatorname{div}(A\Phi) = 0, \qquad \partial_t(A\Phi) + \operatorname{div}(B\Phi) = 0.$$

Furthermore there exists a vector $\boldsymbol{\eta} \in \mathbb{R}^{n+1}$ *such that for all* $\phi \in C_c^\infty(\mathbb{R};\mathbb{R})$

$$A\Phi = (\mathbf{a} - \mathbf{b})\, \phi'''\big((\mathbf{x}, t) \cdot \boldsymbol{\eta}\big)$$
$$B\Phi = \tfrac{1}{r}(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b})\, \phi'''\big((\mathbf{x}, t) \cdot \boldsymbol{\eta}\big)$$

where $\Phi(t, \mathbf{x}) := \phi\big((\mathbf{x}, t) \cdot \boldsymbol{\eta}\big)$.

Proof of the Oscillatory lemma, Step 1

- Let $\varphi \in C_c^\infty\big((-1,1) \times B_1(0), [-1,1]\big)$ be a cutoff function which is identically 1 inside $(-\frac{1}{2}, \frac{1}{2}) \times B_{1/2}(0)$.

- Let $\Psi \in C^\infty(\mathbb{R}, \mathbb{R})$ be defined by $\Psi(y) := -\lambda\, N^{-3} \sin(N\, y)$ where $N > 0$ is a large number to be chosen later.

- Define

$$
\begin{aligned}
\Phi(t, \mathbf{x}) &:= \varphi(t, \mathbf{x})\, \Psi\big((\mathbf{x}, t) \cdot \boldsymbol{\eta}\big) \\
\widehat{\Phi}(t, \mathbf{x}) &:= \Psi\big((\mathbf{x}, t) \cdot \boldsymbol{\eta}\big) \\
(\mathbf{m}_{t_0, \mathbf{x}_0}, U_{t_0, \mathbf{x}_0}) &:= (A\Phi, B\Phi) \\
(\widehat{\mathbf{m}}, \widehat{U}) &:= (A\widehat{\Phi}, B\widehat{\Phi})
\end{aligned}
$$

Proof of the Oscillatory lemma, Step 1

Claim: For all $(t, \mathbf{x}) \in (-1,1) \times B_1(0)$

$$\text{dist}\left((\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x}), U_{t_0,\mathbf{x}_0}(t,\mathbf{x})), \sigma_{t_0,\mathbf{x}_0}\right) < \varepsilon.$$

Proof:
We have

$$\begin{aligned}(\widehat{\mathbf{m}}, \widehat{U}) &= \left((\mathbf{a}-\mathbf{b}), (\mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b})\right) \Psi'''\left((\mathbf{x}, t)\cdot\boldsymbol{\eta}\right) \\ &= \left((\mathbf{a}-\mathbf{b}), (\mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b})\right) \lambda \cos\left(N\,(\mathbf{x}, t)\cdot\boldsymbol{\eta}\right) \\ &= p \cos\left(N\,(\mathbf{x}, t)\cdot\eta\right) \in \sigma_{t_0,\mathbf{x}_0}.\end{aligned}$$

It is not difficult to check that

$$\left\|(\mathbf{m}_{t_0,\mathbf{x}_0}, U_{t_0,\mathbf{x}_0}) - \varphi\,(\widehat{\mathbf{m}}, \widehat{U})\right\|_\infty \le c_0\,\frac{1}{N},$$

where $c_0 > 0$ is a suitable constant. We can choose N large such that $c_0\,\frac{1}{N} < \varepsilon$.

Oscillatory lemma

Lemma

For all compact sets $\Gamma \subset (0, T) \times \Omega$ *there exists a constant* $\beta > 0$ *with the following property. For any given* $\mathbf{m} \in X_0$ *there exists a sequence* $(\mathbf{m}_k)_{k\in\mathbb{N}} \subset X_0$ *such that*

$$\mathbf{m}_k \xrightarrow{d} \mathbf{m}$$

$$\liminf_{k\to\infty} \|\mathbf{m}_k\|^2_{L^2(\Gamma)} \geq \|\mathbf{m}\|^2_{L^2(\Gamma)} + \beta \left(c\, r\, |\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)} \right)^2.$$

Proof of the Oscillatory lemma, Step 1

Define

$$n^* := \dim(\mathbb{R}^n \times \mathcal{S}_0^n) + 1 = n + \sum_{i=1}^{n} i - 1 + 1 = \frac{n(n+3)}{2}.$$

Fix an arbitrary point $(t_0, \mathbf{x}_0) \in \Gamma$. For convenience we define

$$(\mathbf{m}^\star, U^\star) := \big(\mathbf{m}(t_0, \mathbf{x}_0), U(t_0, \mathbf{x}_0)\big).$$

By assumption it holds that $(\mathbf{m}^\star, U^\star) \in \mathcal{U}$.

Proof of the Oscillatory lemma, Step 1

Claim: $\exists$ a segment $\sigma_{t_0,\mathbf{x}_0} = [-p, p] \subset \mathbb{R}^n \times \mathcal{S}_0^n$ such that:

1. $\exists\ \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $|\mathbf{a}| = |\mathbf{b}| = \sqrt{r\,c}$ and $\mathbf{a} \neq \pm\mathbf{b}$, and $\lambda > 0$ such that
$$p = \lambda \left[\left(\mathbf{a}, \frac{\mathbf{a}\otimes\mathbf{a}}{r} \right) - \left(\mathbf{b}, \frac{\mathbf{b}\otimes\mathbf{b}}{r} \right) \right].$$
2. $(\mathbf{m}^\star, U^\star) + \sigma_{t_0,\mathbf{x}_0} \subset \mathcal{U}$.
3. $\forall\ \varepsilon > 0\ \exists$ a pair $(\mathbf{m}_{t_0,\mathbf{x}_0}, U_{t_0,\mathbf{x}_0}) \in C_c^\infty\big((-1,1) \times B_1(0)\big)$ s. t.
   - $\operatorname{div} \mathbf{m}_{t_0,\mathbf{x}_0} = 0 \qquad \partial_t \mathbf{m}_{t_0,\mathbf{x}_0} + \operatorname{div} U_{t_0,\mathbf{x}_0} = 0$
   - $\operatorname{dist}\big((\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x}), U_{t_0,\mathbf{x}_0}(t,\mathbf{x})), \sigma_{t_0,\mathbf{x}_0} \big) < \varepsilon$ for all $(t,\mathbf{x}) \in (-1,1) \times B_1(0)$
   - $\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left| \mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x}) \right| \,d\mathbf{x}\,dt \geq c_1 \left(r\,c - \left| \mathbf{m}(t_0,\mathbf{x}_0) \right|^2 \right)$ for a suitable constant $c_1 > 0$
   - $\int_{\mathbb{R}^n} \mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x})\,d\mathbf{x} = 0.$

Proof of the Oscillatory lemma, Step 2

- Since $(\mathbf{m}, U)$ is uniformly continuous, there exists $\delta_1 > 0$ s. t.

$$\big(\mathbf{m}(t,\mathbf{x}), U(t,\mathbf{x})\big) + \sigma_{t_0,\mathbf{x}_0} \subset \mathcal{U}$$

  for all $(t,\mathbf{x}), (t_0,\mathbf{x}_0) \in \Gamma$ with $|t - t_0| \leq \delta_1$ and $|\mathbf{x} - \mathbf{x}_0| \leq \delta_1$.
- Step 1 yields a pair $(\mathbf{m}_{t_0,\mathbf{x}_0}, U_{t_0,\mathbf{x}_0}) \in C_c^\infty\big((-1,1) \times B_1(0)\big)$ that fulfills

$$\mathrm{dist}\,\big((\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x}), U_{t_0,\mathbf{x}_0}(t,\mathbf{x})), \sigma_{t_0,\mathbf{x}_0}\big) < \varepsilon$$

  for all $(t,\mathbf{x}) \in (-1,1) \times B_1(0)$.
- Define

$$\big(\mathbf{m}_{t_0,\mathbf{x}_0,\delta}, U_{t_0,\mathbf{x}_0,\delta}\big)(t,\mathbf{x}) := (\mathbf{m}_{t_0,\mathbf{x}_0}, U_{t_0,\mathbf{x}_0})\left(\frac{t - t_0}{\delta}, \frac{\mathbf{x} - \mathbf{x}_0}{\delta}\right),$$

  then $\mathrm{supp}\,\big(\mathbf{m}_{t_0,\mathbf{x}_0,\delta}, U_{t_0,\mathbf{x}_0,\delta}\big) \subset (t_0 - \delta, t_0 + \delta) \times B_\delta(\mathbf{x}_0)$.

Proof of the Oscillatory lemma, Step 2

- Additionally we get that

$$\operatorname{dist}\left((\mathbf{m}_{t_0,\mathbf{x}_0,\delta}(t,\mathbf{x}), U_{t_0,\mathbf{x}_0,\delta}(t,\mathbf{x})), \sigma_{t_0,\mathbf{x}_0}\right) < \varepsilon$$

  for all $(t,\mathbf{x}) \in (t_0-\delta, t_0+\delta) \times B_\delta(\mathbf{x}_0)$.

- Because $\mathcal{U}$ is open, we can choose ε so small that

$$\big(\mathbf{m}(t,\mathbf{x}), U(t,\mathbf{x})\big) + \big(\mathbf{m}_{t_0,\mathbf{x}_0,\delta}(t,\mathbf{x}), U_{t_0,\mathbf{x}_0,\delta}(t,\mathbf{x})\big) \in \mathcal{U}$$

  for all $(t,\mathbf{x}) \in (t_0-\delta, t_0+\delta) \times B_\delta(\mathbf{x}_0)$.

- We obtain from step 1 that

$$\begin{aligned}\iint_\Gamma \left|\mathbf{m}_{t_0,\mathbf{x}_0,\delta}(t,\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \delta^{n+1} \int_{-1}^{1}\int_{B_1(0)} \left|\mathbf{m}_{t_0,\mathbf{x}_0}(t,\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\geq \delta^{n+1}\, c_1 \left(r\,c - \left|\mathbf{m}(t_0,\mathbf{x}_0)\right|^2\right).\end{aligned}$$

Proof of the Oscillatory lemma, Final step

There exists a radius $\delta_2 > 0$ and a constant $c_2 > 0$ such that for all $0 < \delta < \delta_2$ there are finitely many points $(t_j, \mathbf{x}_j) \in \Gamma$ with the following properties:

- The sets $(t_j - \delta, t_j + \delta) \times B_\delta(\mathbf{x}_j)$ are contained in Γ and pairwise disjoint.
- The inequality

$$\begin{aligned} &\delta^{n+1} \sum_j \left(r\,c - \left|\mathbf{m}(t_j, \mathbf{x}_j)\right|^2 \right) \\ &\geq c_2 \iint_\Gamma \left(r\,c - \left|\mathbf{m}(t, \mathbf{x})\right|^2 \right) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &= c_2 \left(r\,c\,|\Gamma| - \iint_\Gamma \left|\mathbf{m}(t, \mathbf{x})\right|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \right) \end{aligned}$$

  holds.

Proof of the Oscillatory lemma, Final step

- Let $\delta = \frac{1}{k}$ for $k \in \mathbb{N}$ such that $\frac{1}{k} < \min\{\delta_1, \delta_2\}$.
- Find finitely many points $(t_j, \mathbf{x}_j) \in \Gamma$ as above.
- Do the construction of step 1, 2 for each $(t_j, \mathbf{x}_j)$.
- Define $(\mathbf{m}_k, U_k) = (\mathbf{m}, U) + \sum_j (\mathbf{m}_{t_j, x_j, \delta}, U_{t_j, x_j, \delta})$.

Claim: $\mathbf{m}_k \in X_0$.

Claim: $\mathbf{m}_k \overset{d}{\to} \mathbf{m}$.

Proof of the Oscillatory lemma, Final step

Additionally we have the following estimate

$$
\begin{aligned}
\|\mathbf{m}_k - \mathbf{m}\|_{L^1(\Gamma)} &= \iint_\Gamma \left|\mathbf{m}_k(t,\mathbf{x}) - \mathbf{m}(t,\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \iint_\Gamma \Big|\sum_j \mathbf{m}_{t_j,\mathbf{x}_j,\delta}(t,\mathbf{x})\Big| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \sum_j \iint_\Gamma \left|\mathbf{m}_{t_j,\mathbf{x}_j,\delta}(t,\mathbf{x})\right| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\geq \sum_j \delta^{n+1}\, c_1 \left(r\,c - \left|\mathbf{m}(t_j,\mathbf{x}_j)\right|^2\right) \\
&\geq c_1\, c_2 \left(r\,c\,|\Gamma| - \iint_\Gamma \left|\mathbf{m}(t,\mathbf{x})\right|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right) \\
&= c_1\, c_2 \left(r\,c\,|\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)}\right).
\end{aligned}
$$

Proof of the Oscillatory lemma, Final step

Additionally we have the following estimate

$$\|\mathbf{m}_k - \mathbf{m}\|_{L^1(\Gamma)} \geq c_1\, c_2 \left(r\, c\, |\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)} \right).$$

Furthermore

$$\|\mathbf{m}_k - \mathbf{m}\|_{L^1(\Gamma)} \leq \|\mathbf{m}_k - \mathbf{m}\|_{L^2(\Gamma)}\ \sqrt{|\Gamma|},$$

and therefore

$$\frac{1}{|\Gamma|}\|\mathbf{m}_k - \mathbf{m}\|^2_{L^1(\Gamma)} \leq \|\mathbf{m}_k - \mathbf{m}\|^2_{L^2(\Gamma)}.$$

Putting the previous inequalities together we obtain

$$\|\mathbf{m}_k - \mathbf{m}\|^2_{L^2(\Gamma)} \geq \frac{c_1^2\, c_2^2}{|\Gamma|} \left(r\, c\, |\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)} \right)^2.$$

Proof of the Oscillatory lemma, Final step

Hence

$$
\begin{aligned}
\|\mathbf{m}_k\|^2_{L^2(\Gamma)} &= \|\mathbf{m} + \mathbf{m}_k - \mathbf{m}\|^2_{L^2(\Gamma)} \\
&= \|\mathbf{m}\|^2_{L^2(\Gamma)} + \|\mathbf{m}_k - \mathbf{m}\|^2_{L^2(\Gamma)} \\
&\qquad + 2 \iint_\Gamma \mathbf{m}(t,\mathbf{x}) \left(\mathbf{m}_k(t,\mathbf{x}) - \mathbf{m}(t,\mathbf{x})\right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\geq \|\mathbf{m}\|^2_{L^2(\Gamma)} + \frac{c_1^2 \, c_2^2}{|\Gamma|} \left(r \, c \, |\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)} \right)^2 \\
&\qquad + 2 \iint_\Gamma \mathbf{m}(t,\mathbf{x}) \left(\mathbf{m}_k(t,\mathbf{x}) - \mathbf{m}(t,\mathbf{x})\right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t.
\end{aligned}
$$

Since $\mathbf{m}_k \overset{d}{\to} \mathbf{m}$, the integral tends to 0 as $k \to \infty$.

$$
\liminf_{k\to\infty} \|\mathbf{m}_k\|^2_{L^2(\Gamma)} \geq \|\mathbf{m}\|^2_{L^2(\Gamma)} + \frac{c_1^2 \, c_2^2}{|\Gamma|} \left(r \, c \, |\Gamma| - \|\mathbf{m}\|^2_{L^2(\Gamma)} \right)^2 .
$$

Outline

2-d Compressible Euler equations

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{v}) = 0$$

$$\partial_t(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho\, e(\varrho, p) \right) + \operatorname{div} \left[\left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho\, e(\varrho, p) + p \right) \mathbf{v} \right] = 0$$

Unknowns:

- density $\varrho = \varrho(t, \mathbf{x}) \in \mathbb{R}^+$
- velocity $\mathbf{v} = \mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^2$
- pressure $p = p(t, \mathbf{x}) \in \mathbb{R}^+$

Variables:

- time $t \in [0, \infty)$
- spatial variable $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$

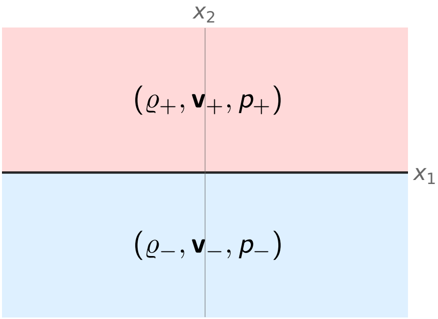
Ideal gas:
$e(\varrho, p) = \frac{1}{\gamma - 1} \frac{p}{\varrho}$, where $\gamma < 3$.

Riemann initial data

We consider

$$(\varrho, \mathbf{v}, p)(0, x) = (\varrho^0, \mathbf{v}^0, p^0)(x) := \begin{cases} (\varrho_-, \mathbf{v}_-, p_-) & \text{if } x_2 < 0 \\ (\varrho_+, \mathbf{v}_+, p_+) & \text{if } x_2 > 0 \end{cases},$$

where $\varrho_\pm \in \mathbb{R}^+$, $\mathbf{v}_\pm \in \mathbb{R}^2$ and $p_\pm \in \mathbb{R}^+$ are constant and $v_{-,1} = v_{+,1} = 0$.



Corresponding 1-d Riemann problem

Solve the corresponding 1-d Riemann problem

$$\partial_t \varrho + \partial_{x_2}(\varrho\, v_2) = 0,$$
$$\partial_t(\varrho\, v_2) + \partial_{x_2}\left(\varrho\, v_2^2 + p\right) = 0,$$
$$\partial_t\left(\frac{1}{2}\varrho\, v_2^2 + \varrho\, e(\varrho, p)\right) + \partial_{x_2}\left[\left(\frac{1}{2}\varrho\, v_2^2 + \varrho\, e(\varrho, p) + p\right) v_2\right] = 0,$$

$$(\varrho, v_2, p)(0, x) = (\varrho^0, v_2^0, p^0)(x) := \begin{cases} (\varrho_-, v_{-,2}, p_-) & \text{if } x_2 < 0 \\ (\varrho_+, v_{+,2}, p_+) & \text{if } x_2 > 0 \end{cases}.$$

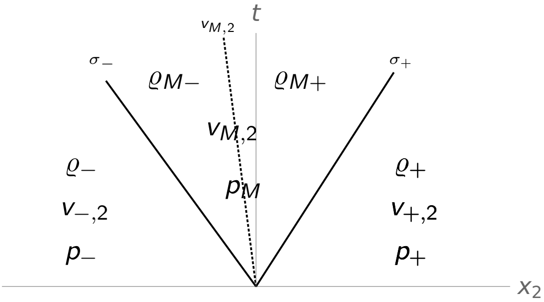
J. Smoller. *Shock waves and reaction-diffusion equations*. New York: Springer-Verlag, 1967

C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*. 4th ed. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg: Springer-Verlag, 2016

Solution of the corresponding 1-d Riemann problem

Constant states seperated by three waves

- 1-wave: Either a shock or a rarefaction wave
- 2-wave: Contact discontinuity
- 3-wave: Either a shock or a rarefaction wave



Result

Theorem

We assume that the initial data $\varrho_\pm \in \mathbb{R}^+$, $\mathbf{v}_\pm \in \mathbb{R}^2$, $p_\pm \in \mathbb{R}^+$ *fulfill* $v_{-,1} = v_{+,1} = 0$ *and are such that the 1-d self-similar solution consists of*

- *a 1-shock, a 2-contact discontinuity and a 3-shock or*
- *a 1-shock and a 3-shock.*

Then there exist infinitly many entropy solutions.

Basic ideas of the non-uniqueness proof

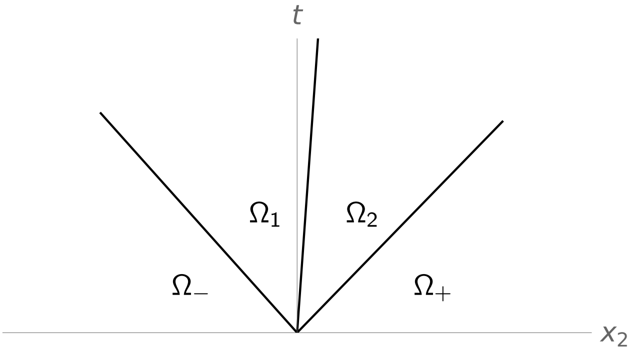
- Definition: fan partition

Let $\mu_0 < \mu_1 < \mu_2$ real numbers. A fan partition of $(0, \infty) \times \mathbb{R}^2$ is a set of 4 open sets $\Omega_-, \Omega_1, \Omega_2, \Omega_+$ of the form

$$
\begin{aligned}
\Omega_- &= \{(t, \mathbf{x}) : t > 0 \text{ and } x_2 < \mu_0\, t\}; \\
\Omega_1 &= \{(t, \mathbf{x}) : t > 0 \text{ and } \mu_0\, t < x_2 < \mu_1\, t\}; \\
\Omega_2 &= \{(t, \mathbf{x}) : t > 0 \text{ and } \mu_1\, t < x_2 < \mu_2\, t\}; \\
\Omega_+ &= \{(t, \mathbf{x}) : t > 0 \text{ and } x_2 > \mu_2\, t\}.
\end{aligned}
$$

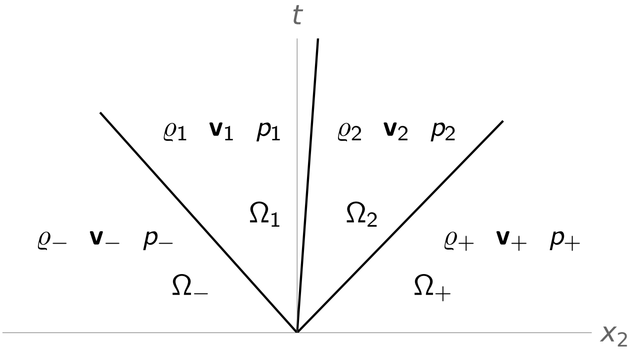
Basic ideas of the non-uniqueness proof

- Definition: fan partition



Basic ideas of the non-uniqueness proof

- Definition: fan partition
- Define a piecewise constant *fan subsolution* $(\overline{\varrho}, \overline{\mathbf{v}}, \overline{p})$



Basic ideas of the non-uniqueness proof

- Definition: fan partition
- Define a piecewise constant *fan subsolution* $(\overline{\varrho}, \overline{\mathbf{v}}, \overline{p})$
- Apply convex integration on Ω_1, Ω_2 to obtain $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$

Proposition

Let $(\widetilde{\mathbf{v}}, \widetilde{U}) \in \mathbb{R}^2 \times \mathcal{S}_0^2$ and $c > 0$ such that $\widetilde{\mathbf{v}} \otimes \widetilde{\mathbf{v}} - \widetilde{U} < \frac{c}{2}\,\mathbb{I}_2$. Furthermore let $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ open. Then there exist infinitely many maps $(\underline{\mathbf{v}}, \underline{U}) \in L^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2 \times \mathcal{S}_0^2)$ with the following properties.

- $\underline{\mathbf{v}}$ and $\underline{U}$ vanish outside Ω.
- $\operatorname{div} \underline{\mathbf{v}} = 0$ and $\partial_t \underline{\mathbf{v}} + \operatorname{div} \underline{U} = 0$ in the sense of distributions.
- $(\widetilde{\mathbf{v}} + \underline{\mathbf{v}}) \otimes (\widetilde{\mathbf{v}} + \underline{\mathbf{v}}) - (\widetilde{U} + \underline{U}) = \frac{c}{2}\,\mathbb{I}_2$ a.e. on Ω.

C. De Lellis E. Chiodaroli and O. Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: *Comm. Pure Appl. Math.* 68.7 (2015), pp. 1157–1190

Basic ideas of the non-uniqueness proof

- Definition: fan partition
- Define a piecewise constant *fan subsolution* $(\overline{\varrho}, \overline{\mathbf{v}}, \overline{p})$
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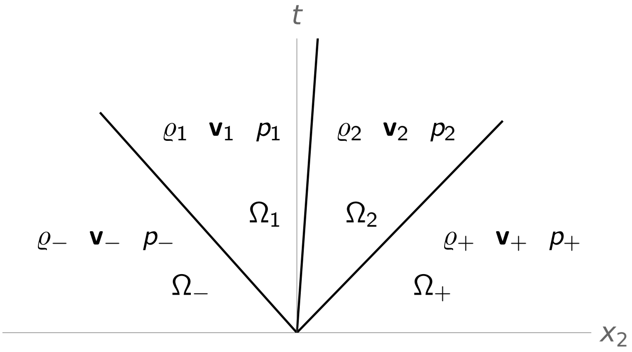
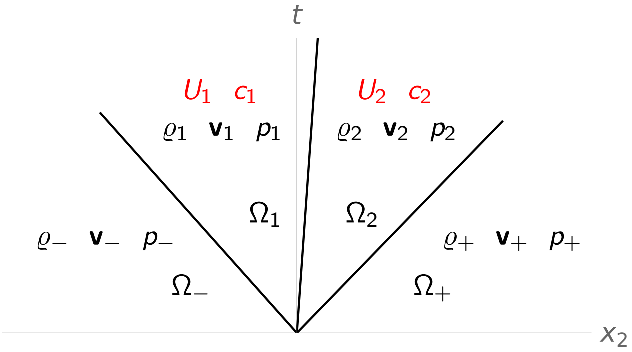


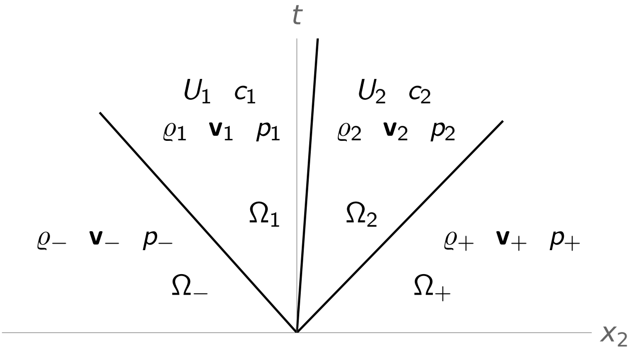
Basic ideas of the non-uniqueness proof

- Definition: fan partition
- Define a piecewise constant *fan subsolution* $(\overline{\varrho}, \overline{\mathbf{v}}, \overline{p}, \overline{U}, \overline{c})$
- Apply convex integration on Ω_1, Ω_2 to obtain $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$



Basic ideas of the non-uniqueness proof

- Definition: fan partition
- Define a piecewise constant *fan subsolution* $(\overline{\varrho}, \overline{\mathbf{v}}, \overline{p}, \overline{U}, \overline{c})$
- Apply convex integration on Ω_1, Ω_2 to obtain $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$
- Define the fan subsolution such that $(\overline{\varrho}\,,\, \overline{\mathbf{v}} + \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2\,,\, \overline{p})$ is a solution



Definition: admissible fan subsolution (1)

An adm. fan subsolution consists of 5 piecewise constant functions $(\overline{\varrho}, \overline{\mathbf{v}}, \overline{p}, \overline{U}, \overline{c}) : (0, \infty) \times \mathbb{R}^2 \to (\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathcal{S}_0^2 \times \mathbb{R}^+)$, which satisfy the following properties:

- There exists a fan partition $\Omega_-, \Omega_1, \Omega_2, \Omega_+$ such that

$$(\overline{\varrho}, \overline{\mathbf{v}}, \overline{p}, \overline{U}, \overline{c}) = \begin{cases} \big(\varrho_\pm\,,\, \mathbf{v}_\pm\,,\, p_\pm\,,\, U_\pm\,,\, c_\pm\big) & \text{on } \Omega_\pm \\ \big(\varrho_1\,,\, \mathbf{v}_1\,,\, p_1\,,\, U_1\,,\, c_1\big) & \text{on } \Omega_1 \\ \big(\varrho_2\,,\, \mathbf{v}_2\,,\, p_2\,,\, U_2\,,\, c_2\big) & \text{on } \Omega_2 \end{cases}$$

  where $U_\pm = \mathbf{v}_\pm \otimes \mathbf{v}_\pm - \frac{1}{2}|\mathbf{v}_\pm|^2\,\mathbb{I}_2$ and $c_\pm = |\mathbf{v}_\pm|^2$.
- The following inequalities hold in the sense of definiteness

$$\mathbf{v}_1 \otimes \mathbf{v}_1 - U_1 < \tfrac{1}{2} c_1\,\mathbb{I}_2, \qquad \mathbf{v}_2 \otimes \mathbf{v}_2 - U_2 < \tfrac{1}{2} c_2\,\mathbb{I}_2.$$

Definition: admissible fan subsolution (2)

- The following identities hold in the sense of distributions:

$$\begin{aligned}
\partial_t \overline{\varrho} + \mathrm{div}(\overline{\varrho}\, \overline{\mathbf{v}}) &= 0, \\
\partial_t (\overline{\varrho}\, \overline{\mathbf{v}}) + \mathrm{div}(\overline{\varrho}\, \overline{U}) + \nabla \left(\overline{p} + \tfrac{1}{2} \overline{\varrho}\, \overline{c} \right) &= 0, \\
\partial_t \left(\tfrac{1}{2}\, \overline{\varrho}\, \overline{c} + \tfrac{1}{\gamma - 1}\, \overline{p} \right) + \mathrm{div} \left[\left(\tfrac{1}{2}\, \overline{\varrho}\, \overline{c} + (\tfrac{1}{\gamma-1} + 1)\, \overline{p} \right) \overline{\mathbf{v}} \right] &= 0.
\end{aligned}$$

- The entropy inequality is fulfilled in the sense of distributions:

$$\partial_t \left(\overline{\varrho}\, s(\overline{\varrho}, \overline{p}) \right) + \mathrm{div} \left(\overline{\varrho}\, s(\overline{\varrho}, \overline{p})\, \overline{\mathbf{v}} \right) \quad \geq \quad 0$$

Condition for the existence of infinitely many solutions

Proposition

Existence of an admissible fan subsolution $\Longrightarrow$ *Existence of infinitely many entropy solutions*

Results

1-wave	2-wave	3-wave	1-wave	2-wave	3-wave
-	-	-	-	contact	-
-	-	shock	-	contact	shock
-	-	raref.	-	contact	raref.
shock	-	-	shock	contact	-
shock	-	shock	shock	contact	shock
shock	-	raref.	shock	contact	raref.
raref.	-	-	raref.	contact	-
raref.	-	shock	raref.	contact	shock
raref.	-	raref.	raref.	contact	raref.

Question

Is the 1-d self-similar solution the *unique* entropy solution to the 2-d problem?

Results

1-wave	2-wave	3-wave	1-wave	2-wave	3-wave
-	-	-	-	contact	-
-	-	shock	-	contact	shock
-	-	raref.	-	contact	raref.
shock	-	-	shock	contact	-
shock	-	shock	shock	contact	shock
shock	-	raref.	shock	contact	raref.
raref.	-	-	raref.	contact	-
raref.	-	shock	raref.	contact	shock
raref.	-	raref.	raref.	contact	raref.

non-unique

O. Kreml V. Mácha H. Al Baba C. Klingenberg and S. Markfelder. "Non-uniqueness of admissible weak solutions to the Riemann problem for the full Euler system in 2D". In: *submitted* (2018). arXiv: 1805.11354

Results

1-wave	2-wave	3-wave	1-wave	2-wave	3-wave
-	-	-	-	contact	-
-	-	shock	-	contact	shock
-	-	raref.	-	contact	raref.
shock	-	-	shock	contact	-
shock	-	shock	shock	contact	shock
shock	-	raref.	shock	contact	raref.
raref.	-	-	raref.	contact	-
raref.	-	shock	raref.	contact	shock
raref.	-	raref.	raref.	contact	raref.

non-unique

unique

G.-Q. Chen and J. Chen. "Stability of rarefaction waves and vacuum states for the multidimensional Euler equations". In: *J. Hyperbolic Differ. Equ.* 4.1 (2007), pp. 105–122

Results for the isentropic Euler equations

1-wave	2-wave
-	-
-	shock
-	raref.
shock	-
shock	shock
shock	raref.
raref.	-
raref.	shock
raref.	raref.

Question

Is the 1-d self-similar solution the *unique* entropy solution to the 2-d problem?

Results for the isentropic Euler equations

1-wave	2-wave
-	-
-	shock
-	raref.
shock	-
shock	shock
shock	raref.
raref.	-
raref.	shock
raref.	raref.

non-unique
unique

References:

G.-Q. Chen and J. Chen. "Stability of rarefaction waves and vacuum states for the multidimensional Euler equations". In: *J. Hyperbolic Differ. Equ.* 4.1 (2007), pp. 105–122

E. Chiodaroli and O. Kreml. "On the energy dissipation rate of solutions to the compressible isentropic Euler system". In: *Arch. Ration. Mech. Anal.* 214.3 (2014), pp. 1019–1049

C. Klingenberg and S. Markfelder. "The Riemann problem for the multidimensional isentropic system of gas dynamics is ill-posed if it contains a shock". In: *Arch. Rational Mech. Anal.* 227.3 (2018), pp. 967–994

Outline