Onsager's conjecture

[In these lectures we consider the following system of PDE] Incompressible Euler Equations:

$$(E) \quad \begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 & \text{in } \mathbb{T}^3 \times [0,T] \\ \operatorname{div} v = 0 & \text{in } \mathbb{T}^3 \times [0,T] \end{cases}$$

where $v : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3$ velocity, $p : \mathbb{T}^3 \times [0,T] \to \mathbb{R}$ pressure

[Motion of an ideal, incompressible fluid, periodic boundary conditions. (incompressible preserves area)]

Classical solutions $(v, p \in C^1(\mathbb{T}^3 \times [0,T]))$

Theorem: Let $v_0 \in C^{1,\gamma}(\mathbb{T}^3)$, $\operatorname{div} v_0 = 0$.
Then $\exists\ T = T(\|v_0\|_{C^{1,\gamma}}) > 0$ and $(v,p) : \mathbb{T}^3 \times [0,T] \longrightarrow \mathbb{R}^3 \times \mathbb{R}$ classical solution of (E) with $v(0,\cdot) = v_0$.

[Global in time solutions: long-standing difficult open problem.]

Conservation of the energy

Let $\quad E(t) = \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx \qquad \forall\, t \in [0,T]$

$$\frac{d}{dt} E(t) = \int_{\mathbb{T}^3} v \cdot \partial_t v \, dx = \int_{\mathbb{T}^3} - v \operatorname{div}(v \otimes v) - \int_{\mathbb{T}^3} v \nabla p$$

$$= - \int_{\mathbb{T}^3} v_i v_j \partial_j v_i - \int_{\mathbb{T}^3} \overset{\overset{0}{\parallel}}{\operatorname{div} v} \cdot p$$

$$= \int_{\mathbb{T}^3} \partial_j v_i (v_j v_i) + v_i^2 \overset{\overset{0}{\parallel}}{\partial_j v_j}$$

$$= 0$$

Formally, the Euler equations can be obtained as the inviscid limit of the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \nu \Delta v \\ \operatorname{div} v = 0 \end{cases}$$

[ν is the viscosity]

For N.-S.

$$\frac{d}{dt} E(t) = -\nu \int |\nabla v|^2 < 0 \quad \Rightarrow \text{dissipation.}$$

What is observed in experiments is that, for arbitrarily small viscosity, dissipation persists. Correspondingly then, the L^2-norm of the gradient would have to blow-up.

Onsager '49 conjectured that there may exist solutions of the Euler equations, non-classical, which dissipate the total kinetic energy.

(*) Def.: We say $v \in C^\alpha$, $\alpha \in (0,1)$ if $\exists C > 0$ s.t.
$\forall x, y \in \mathbb{T}^3$, $\forall t \in [0,T]$

$$|v(x,t) - v(y,t)| \leq C|x-y|^\alpha$$

Conjecture: If $v \in C^\alpha$ weak solution of (E) and

(1) $\alpha > \frac{1}{3}$, then energy must be conserved;

(2) $\alpha < \frac{1}{3}$, then energy might be dissipated.

(*) [Def. (weak solution of (E)) A function $v \in L^\infty([0,T]; L^2(\mathbb{T}^3))$ is a weak solution of (E) if, $\forall \psi \in C_c^\infty(\mathbb{T}^3 \times (0,T))$, $\operatorname{div} \psi = 0$ and $\forall \varphi \in C_c^\infty(\mathbb{T}^3 \times (0,T))$,

$$\int_0^T \int_{\mathbb{T}^3} v \, \partial_t \psi + (v \otimes v) : \nabla \psi = 0$$

$$\int_0^T \int_{\mathbb{T}^3} v \cdot \nabla \varphi = 0 \quad]$$

Part (1) (Constantin, E, Titi '94)

Theorem: Let $v \in C^\alpha$ be a weak solution of (E). If $\alpha > \frac{1}{3}$, then $E(t) = E(0)$ $\forall\, t \in [0,T]$.

Proof: Let $\{\varphi_\varepsilon\}$ be a family of standard mollification kernels on $\mathbb{T}^3$. Let $v_\varepsilon = v * \varphi_\varepsilon$. $v_\varepsilon \in C^\infty(\mathbb{T}^3)$

eq. for v_ε →

$$\frac{d}{dt}\int |v_\varepsilon|^2 = \int v_\varepsilon \partial_t v_\varepsilon = \int (v\otimes v)_\varepsilon : \nabla v_\varepsilon = \int \mathrm{Tr}\left((v\otimes v)_\varepsilon \nabla v_\varepsilon\right)$$

$$\int |v_\varepsilon(x,t)|^2\,dx - \int |v_\varepsilon(x,0)|^2 = \int_0^t \int_{\mathbb{T}^3} \mathrm{Tr}\left((v\otimes v)_\varepsilon \nabla v_\varepsilon\right) dx\,dt$$

Aim: To show $\displaystyle\int_0^t \int_{\mathbb{T}^3} \mathrm{Tr}\left((v\otimes v)_\varepsilon \nabla v_\varepsilon\right) dx\,d\tau \underset{\varepsilon\to 0}{\longrightarrow} 0$

$$\int_{\mathbb{T}^3} \mathrm{Tr}\left(v_\varepsilon \otimes v_\varepsilon \nabla v_\varepsilon\right) = 0.$$

$$\boxed{(v\otimes v)_\varepsilon = v_\varepsilon \otimes v_\varepsilon + r_\varepsilon(v,v) - (v-v_\varepsilon)\otimes(v-v_\varepsilon)}$$ ← discuss

where $r_\varepsilon(v,v) = \int \varphi_\varepsilon(y)\left[(v(x-y)-v(x))\otimes(v(x-y)-v(x))\right] dy$

$$\|r_\varepsilon(v,v)\|_{L^{3/2}} \le C\int_{B_\varepsilon(0)} \|(v(\cdot-y)-v(\cdot))^2\|_{L^{3/2}}\,dy \le C\varepsilon^{2\alpha}\|v\|^2_{C^\alpha}$$

$$\|(v-v_\varepsilon)^2\|_{L^{3/2}} \le C\varepsilon^{2\alpha}\|v\|^2_{C^\alpha}, \quad \|\nabla v_\varepsilon\|_{L^3} \le C\varepsilon^{\alpha-1}\|v\|_{C^\alpha}$$

$$\left|\int |v_\varepsilon(x,t)|^2 dx - \int |v_\varepsilon(x,0)|^2 dx\right| \le C\int_0^t d\tau \Big\{ \|r_\varepsilon\|_{L^{3/2}} \|\nabla v_\varepsilon\|_{L^3}$$
$$+ \|(v-v_\varepsilon)^2\|_{L^{3/2}}\|\nabla v_\varepsilon\|_{L^3}\Big\} \le C\varepsilon^{\alpha-1/3}\int_0^t \|v_\varepsilon\|^3_{C^\alpha} \to 0$$

if $\alpha > \frac{1}{3}$.

Part (2)

- Scheffer, Shnirelman '90s
Construction of compactly supported L^2 (and then L^∞) weak solutions in 2d and 3d. ($\mathbb{T}^3 \leftrightarrow D \subset \mathbb{R}^d$)

- De Lellis and Székelyhidi '07
$\exists\ \infty$ many L^∞ compactly supported solutions in arbitrary dimensions ($\mathbb{T}^3 \leftrightarrow D$)

- De Lellis and Székelyhidi '10
Given $e \in C([0,T]; \mathbb{R}^+)$, there exist ∞ many L^∞ ~~compactly supported~~ solutions of (E) with total kinetic energy e. ($\mathbb{T}^3 \leftrightarrow D$)

- De Lellis and Szekelyhidi '12
Given $e \in C^\infty([0,T]; \mathbb{R}^+)$, there exist ∞ many C^0 solutions of (E) with $\int |v|^2(t) = e(t)$.

- De Lellis and Szekelyhidi '12
C^α, $\alpha < \frac{1}{10}$

- Isett '14
$\exists$ of compactly supported solutions in C^α, $\alpha < \frac{1}{5}$ (preassigned total kinetic energy by Buckmaster, De Lellis, Isett, Szek.)

- Daneri, Szekelyhidi '16
Non-uniqueness of C^α solutions, $\alpha < \frac{1}{5}$ and density in L^2 of non-uniqueness initial data

AIM: To give overview of the proof of [BDSV] $\longrightarrow$
- Isett '16 $\exists$ of compactly supp. sol. in C^α, $\alpha < \frac{1}{3}$
(preass. tot. kin. en. by Buck. De L. Sz. and Vicol '17)

Strategy of the proof (starting from De Lellis and Szekelyhidi '07)

CONVEX INTEGRATION

1) Start from a SUBSOLUTION of the problem
2) Add iteratively a sequence of nonlinear perturbations to the flow so that at each step one obtains still a subsolution, but with a smaller gap from being a solution.
3) At each step obtain estimates on the flow that imply convergence of the subsolutions to a solution of the problem in the desired topology.

[Convex integration for the first time apply by Nash for the problem of isometric immersions. Let us look at a toy example ...]

1d isometric immersions in $\mathbb{R}^3$

Look for $\gamma \in C^0([0,1];\mathbb{R}^3)$ s.t. $|\dot\gamma| = 1$ and $\gamma([0,1]) \subset B_\varepsilon(0) \subset \mathbb{R}^3$.

Idea: obtain γ as the C^0-limit of $\{\gamma_n\} \subset C^\infty([0,1]; B_{\varepsilon/2}(0))$ with

subsolutions

$$1 - \delta_n < |\dot\gamma_n|^2 < 1, \quad \{\delta_n\} \subset (0,1), \quad \delta_n \downarrow 0$$

$$\gamma_{n+1}(t) = \gamma_n(t) + \frac{\sqrt{\delta_n - \delta_{n+1}}}{\sqrt{2}\,\lambda_{n+1}} \cos(\lambda_{n+1} t)\,\vec{n}$$

Picture

$$+ \frac{\sqrt{\delta_n - \delta_{n+1}}}{\sqrt{2}\,\lambda_{n+1}} \sin(\lambda_{n+1} t)\,\vec{b}$$

where $\lambda_{n+1} \gg 1$, and $\frac{\dot\gamma_n}{|\dot\gamma_n|}, \vec{b}, \vec{n}$ regularized Frenet frame

$$|\gamma_{n+1} - \gamma_n| \sim \frac{\sqrt{\delta_n}}{\lambda_{n+1}}, \quad |\dot\gamma_{n+1} - \dot\gamma_n| \sim \sqrt{\delta_n}$$

$$|\dot\gamma_{n+1}|^2 = |\dot\gamma_n|^2 + \delta_n - \delta_{n+1} + O(\lambda_{n+1}^{-1}).$$

Subsolutions for the Euler equations

Euler - Reynolds system

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = -\operatorname{div} R \\ \operatorname{div} v = 0 \end{cases}$$

where $R \in S^{3\times 3}_{>0}$.

Typical equation for averages of (E), which is used in turbulence theory (there $\mathring{R} \geq 0$)
We ask also that $R = \rho(t)\,\mathrm{Id} + \mathring{R}$.
Moreover

$$\int_{\mathbb{T}^3} |v(x,t)|^2\,dx < e(t) \qquad \forall t \in [0,T]$$

Let $v_q = v$, $v_{q+1} = v + w$ and let us try to make an Ansätz on the perturbations.

$$\|v_q - v_{q-1}\|_0 \leq \delta_q^{1/2}$$
$$\|v_q\|_1 \leq \delta_q^{1/2} \lambda_q$$

$$\|v_q - v_{q-1}\|_\beta \leq \delta_q^{1/2(1-\beta)} \delta_q^{1/2\,\beta} \lambda_q^\beta \leq \delta_q^{1/2} \lambda_q^\beta$$

$$\delta_q = \lambda_q^{-2\beta}, \quad \xRightarrow{\lambda_q \to +\infty} \quad v_q \text{ converge in } C^{\beta'} \ \forall \beta' < \beta.$$

$$\left[\lambda_q = \left[a^{b^q}\right], \quad a \gg 1, \quad 1 < b < 1+\varepsilon\right]$$

Also $\|R_q\|_0$ and $e(t) - \int |v(t)|^2$ have to converge to 0.

$$\partial_t (v_q + w) + \operatorname{div}(v_q+w)\otimes(v_q+w) + \nabla(p_q + p) =$$

$$= -\operatorname{div} \mathring{R}_{q+1}$$

$$= \operatorname{div}(w \otimes w + p\,\mathrm{Id} - \mathring{R}_q) \quad \text{oscillation term}$$

$$+ \partial_t w + v_q \cdot \nabla w \quad \text{transport term}$$

$$+ w \cdot \nabla v_q \quad \text{Nash error term}$$

In first approximation

$$w(x,t) = \sum_{k \in \mathbb{N}} a_k(x,t) e^{i\lambda k \cdot x} = W(x,t,\lambda x), \quad \lambda >> 1$$

$$W(x,t,\xi) = \sum_{k\in \mathbb{N}} a_k(x,t)\, e^{ik\cdot \xi}$$

Lemma: If $\operatorname{div} w = 0$, $\exists\, \mathring{R}_{q+1} = R\,(\text{o. term} + \text{t. term} + \text{N. term})$

R elliptic operator. Moreover, $\forall\, m \in \mathbb{N}$, $\alpha \in (0,1)$, $\exists\, C = C(m,\alpha)$

s.t. $\forall\ F(x) = a(x,t) e^{i\lambda k\cdot x}$, $k \neq 0$

$$\|R(F)\|_\alpha \le C \left(\frac{\|a\|_0}{\lambda^{1-\alpha}} + \frac{[a]_m}{\lambda^{m-\alpha}} + \frac{[a]_{m+\alpha}}{\lambda^m} \right)$$

$\Big[(Rf)^{ij} = R^{ijk} f^k$, where $R^{ijk} = -\frac{1}{2}\Delta^{-2}\partial_i\partial_j\partial_k + \frac{1}{2}\Delta^{-1}\partial_k \delta_{ij}$

$- \Delta^{-1}\partial_i \delta_{jk} - \Delta^{-1}\partial_j \delta_{ik}$, operator of order $-1 \Big]$.

So, the terms of order λ and higher in the error terms should vanish.

In particular, this gives

$$\begin{cases} \operatorname{div}_\xi W \otimes W = \nabla p \\ \operatorname{div}_\xi W = 0 \end{cases} \quad \leftarrow \text{stationary solutions of Euler}$$

and $$\int W \otimes W \, d\xi = f(t)\, Id + \mathring{R}_q$$

$$w \sim \delta_{q+1}^{1/2} \quad \longrightarrow \quad \boxed{\|\mathring{R}_q\| \leq \delta_{q+1} \left(\lambda_q^{-3\theta}\right)} \qquad 0 < \theta \ll 1$$

(Beltrami flows up to 1/5 paper)

<u>Mikado flows</u> (D., Sz. '16)

<u>Lemma</u>: For any compact subset $\mathcal{N} \subset\subset S^{3\times 3}_+$ there exists a smooth vector field

$$W : \mathcal{N} \times \mathbb{T}^3 \longrightarrow \mathbb{R}^3$$

such that, $\forall R \in \mathcal{N}$

$$\begin{cases} \operatorname{div}_\xi (W(R,\xi) \otimes W(R,\xi)) = 0 \\ \operatorname{div}_\xi W(R,\xi) = 0 \end{cases}$$

$$\fint_{\mathbb{T}^3} W(R,\xi) = 0$$

$$\fint_{\mathbb{T}^3} W(R,\xi) \otimes W(R,\xi) = R.$$

Lemma: $\forall$ compact subset $\mathcal{N} \subset\subset S^{3\times 3}_+$ there exists $\lambda_0 \geq 1$ and smooth functions $\Gamma_k \in C^\infty(\mathcal{N}; [0,1])$ $\forall k \in \mathbb{Z}^3$ with $|k| \leq \lambda_0$ such that

$$R = \sum_{k \in \mathbb{Z}^3, |k| \leq \lambda_0} \Gamma_k^2(R)\, k \otimes k \,, \quad \forall R \in \mathcal{N} .$$

Then we take

$$W(R, \xi) = \sum_{k \in \mathbb{Z}^3, |k| \leq \lambda_0} \Gamma_k(R)\, \psi_k(\xi)\, k$$

where $\psi_k(\xi) = g_k\left(\mathrm{dist}(\xi, \ell_{k,p_k})\right)$ with $g_k \in C^\infty_c([0, r_k))$ $r_k > 0$ and ℓ_{k,p_k} is the $\mathbb{T}^3$-periodic extension of the line $\{p_k + tk : t \in \mathbb{R}\}$ passing through p_k in direction k. One chooses p_k and $r_k > 0$ so that

$$\mathrm{supp}\, \psi_i \cap \mathrm{supp}\, \psi_j = \phi \quad \forall i \neq j .$$

and g_k so that $\quad \fint \psi_k^2(\xi)\, d\xi = 1 \quad \forall k$

Then $$\fint W \otimes W \, d\xi = \sum_k \fint \Gamma_k^2(R)\, \psi_k^2(\xi)\, k \otimes k$$

$$= \sum_k \Gamma_k^2(R)\, k \otimes k = R .$$

Transport error $\quad \partial_t w + v_q \cdot \nabla w$ ← what happens for the previous ansatz → 1/10

We take Phase of the form

$$\sum_{jk} \chi_j e^{i\lambda_{q+1} k\cdot \Phi_j(x,t)} a_{j,k}(x,t)$$

where
$$\begin{cases} \partial_t \Phi_j + v_q \cdot \nabla \Phi_j = 0 \\ \Phi_j(x,t_j) = x \end{cases}$$

$$|t_j - t_{j-1}| \leq \frac{1}{2}[v_q]^{-1}$$

$$\left(\|\nabla \Phi_j - Id\|_0 \leq |t-t_j| [v]_1 \right)$$ ← estimates $\| \|_N$

cut-off necessary in order to have good estimates for the transport equation and stationary lemma.

Nash error $\| \operatorname{div}^{-1}(w \cdot \nabla v_q)\|_0 \sim \dfrac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}$

$$\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} \leq \delta_{q+2}\left(\lambda_{q+1}^{-3\theta}\right)$$

$\rightarrow [\lambda_q = \lceil a^{b^q} \rceil]$

$$a^{b^q(-\alpha b - \alpha + 1 - b)} \leq a^{b^q(-2\alpha b^2)}$$

$$\alpha(2b^2 - b - 1) + 1 - b < 0$$

$$\alpha < \frac{b-1}{(b-1)(2b+1)} = \frac{1}{2b+1} < \frac{1}{3}.$$

Energy estimate What is $f(t)$?

$$\int_{\mathbb{T}_3} |v_q + w|^2 = \int_{\mathbb{T}^3} |v_q|^2 + \int_{\mathbb{T}^3} |w|^2 + 2\int_{\mathbb{T}^3} v_q \cdot w$$

$$\sim \int_{\mathbb{T}^3} |v_q|^2 + \int_{\mathbb{T}^3} 3 f(t)$$

$$= \int_{\mathbb{T}^3} |v_q|^2 + 3(2\pi)^3 f(t)$$

In order to reduce the gap, choose

$$f(t) = \frac{1}{3(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3} |v_q|^2(t) - \delta_{q+2}\, e(t)\right).$$

$$\left(\begin{array}{l} e(t) - \int |v_q|^2 \sim \delta_{q+1}\,, \\ e(t) - \int |v_{q+1}|^2 \sim \delta_{q+2} \end{array}\right)$$

→ Mollification (in order not to have loss of derivatives)

We mollify v_q at length scale ℓ, for some fixed $\ell = \ell(q)$.

$$v_\ell := v_q * \psi_\ell \qquad p_\ell = p_q * \psi_\ell$$

$$\begin{cases} \partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell = \operatorname{div} \mathring{R}_\ell \\ \operatorname{div} v_\ell = 0 \end{cases}$$

where

$$\mathring{R}_\ell = \mathring{R}_q * \psi_\ell + (v_q \mathring{\otimes} v_q) * \psi_\ell - v_\ell \mathring{\otimes} v_\ell$$

$$\|v_\ell\|_{N+1} \lesssim \|v_q\|_1 \ell^{-N}, \quad \|v_q - v_\ell\|_0 \leq \|v_q\|_1 \ell$$

Proposition: Let

$$\ell = \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2} \lambda_q^{1+3\theta/2}} \quad , \quad \theta \in (0,1) \text{ small}$$

Then

$$\|v_\ell - v_q\|_0 \lesssim \delta_{q+1}^{1/2} \lambda_q^{-\theta}$$

$$\|v_\ell\|_{N+1} \lesssim \delta_q^{1/2} \lambda_q \ell^{-N}$$

$$\|\mathring{R}_\ell\|_{N+\theta} \lesssim \delta_{q+1} \ell^{-N+\theta}$$

$$\left| \int |v_q|^2 - |v_\ell|^2 \right| \lesssim \delta_{q+1} \ell^{\theta}$$

Proof: $\|v_\ell - v_q\|_0 \leq \|v_q\|_1 \ell \lesssim \delta_q^{1/2} \lambda_q \ell \lesssim \delta_{q+1}^{1/2} \lambda_q^{-\theta}$

- - -

Problem in the Ansatz from Transport Error

$$\chi_i W_i + \chi_j W_j \qquad \text{for } t \text{ s.t. } \chi_i(t), \chi_j(t) \neq 0.$$

$$\chi_i \chi_j W_i \otimes W_j .$$

Gluing (Isett '16) Aim is to produce a new $\overline{v}_q$, close to v_ℓ (and then to v_q) whose associated Reynolds stress has support in pairwise disjoint temporal regions of length τ_q in time, where τ_q

$$\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q} \ll [v_\ell]^{-1}.$$

For any $n \in \mathbb{Z}$, let

$$t_n = n\tau_q, \qquad I_n = \left[t_n + \tfrac{1}{3}\tau_q, t_n + \tfrac{2}{3}\tau_q\right] \cap [0,T]$$
$$J_n = \left(t_n - \tfrac{1}{3}\tau_q, t_n + \tfrac{1}{3}\tau_q\right) \cap [0,T]$$

We want to build $(\overline{v}_q, \overline{p}_q, \mathring{\overline{R}}_q)$ so that

$$\operatorname{supp} \mathring{\overline{R}}_q \subset \bigcup_{n\in\mathbb{N}} I_n \times \mathbb{T}^3 .$$

$\leftarrow$ general idea

For each n, let $t_n = n\tau_q$ and consider smooth solutions of the Euler equations

$$\begin{cases} \partial_t v_n + \operatorname{div}(v_n \otimes v_n) + \nabla p_n = 0 \\ \operatorname{div} v_n = 0 \\ v_n(\cdot, t_n) = v_\ell(\cdot, t_n) \end{cases}$$

defined over their maximal interval of existence.

From the classical results, we recall the following

<u>Proposition:</u> $\forall \alpha > 0$ $\exists$ $c = c(\alpha) > 0$ with the following property. Given any $v_0 \in C^\infty$ and $T \leq c \|v_0\|_{1+\alpha}^{-1}$, there exists a unique solution $v : \mathbb{T}^3 \times [-T, T] \to \mathbb{R}^3$ to ~~the~~ (E) s.t. $v(\cdot, 0) = v_0$. Moreover, v obeys the bounds

$$\|v\|_{N+\alpha} \lesssim \|v_0\|_{N+\alpha}$$

Apply now Proposition to $v_0 = v_e(\cdot, t_m) \Rightarrow v_m = v$.

Now, let us define a partition of unity $\{\chi_m\}_{m \in \mathbb{N}}$ with the following properties:

- $\sum_m \chi_m \equiv 1$ on $[0, T]$
- $\operatorname{supp} \chi_m \cap \operatorname{supp} \chi_{m+2} = \phi$
  and moreover
  $$\operatorname{supp} \chi_m \subset \left(t_m - \tfrac{2}{3}\tau_q,\ t_m + \tfrac{2}{3}\tau_q\right)$$
  $$\chi_m(t) = 1 \quad \text{for } t \in J_m$$
- $\forall$ m, N
  $$\|\partial_t^N \chi_m\| \leq \tau_q^{-N}$$

Define then

$$\bar{v}_q = \sum_m \chi_m v_m$$

$$\bar{p}_q^{(1)} = \sum_m \chi_m p_m$$

If $t \in I_n$ then $\chi_n + \chi_{n+1} = 1$ and $\chi_j = 0$ for $j \neq n, n+1$.

On I_n

$$\bar v_q = \chi_n v_n + (1-\chi_n) v_{n+1}$$
$$\bar p_q^{(1)} = \chi_n p_n + (1-\chi_n) p_{n+1}$$

and

$$\begin{aligned}
\partial_t \bar v_q + \operatorname{div}(\bar v_q \otimes \bar v_q) + \nabla \bar p_q^{(1)} &= \chi_n \partial_t v_n + (1-\chi_n)\partial_t v_{n+1} \\
&\quad + \partial_t \chi_n (v_n - v_{n+1}) + \operatorname{div}(\chi_n^2 v_n \otimes v_n + \\
&\quad + (1-\chi_n)^2 v_{n+1} \otimes v_{n+1}) \\
&\quad + \chi_n (1-\chi_n) \operatorname{div}(v_n \otimes v_{n+1} + v_{n+1} \otimes v_n) \\
&\quad + \chi_n \nabla p_n + (1-\chi_n) \nabla p_{n+1} \\
&= \partial_t \chi_n (v_n - v_{n+1}) \\
&\quad - \chi_n (1-\chi_n) \operatorname{div}((v_n - v_{n+1}) \otimes (v_n - v_{n+1}))
\end{aligned}$$

On J_n, $\chi_n = 1$ and $\chi_j = 0$ $\forall\, j = n$

Then

$$\bar v_q = v_n \qquad \bar p_q^{(1)} = p_n$$

and

$$\partial_t \bar v_q + \operatorname{div}(\bar v_q \otimes \bar v_q) + \nabla \bar p_q^{(1)} = 0$$

Then, one defines

$$\mathring{\bar R}_q = \partial_t \chi_n \mathcal{R}(v_n - v_{n+1}) - \chi_n (1-\chi_n)(v_n - v_{n+1}) \mathring{\otimes} (v_n - v_{n+1})$$
$$\bar p_q^{(2)} = +\chi_n (1-\chi_n) |v_n - v_{n+1}|^2$$

for $t \in I_n$ and $\mathring{\bar R}_q = 0$, $\bar p_q^{(2)} = 0$ for $t \notin \bigcup_n I_n$, $\bar p_q = \bar p_q^{(1)} + \bar p_q^{(}$

Then, $\mathring{\bar{R}}_q$ is a smooth ~~and~~ symmetric and traceless matrix s.t.

$$\begin{cases} \partial_t \bar{v}_q + \operatorname{div}(\bar{v}_q \otimes \bar{v}_q) + \nabla \bar{p}_q = \operatorname{div} \mathring{\bar{R}}_q \\ \operatorname{div} \bar{v}_q = 0 \end{cases} \quad \text{on } \mathbb{T}^3 \times [0,T].$$

One can show that the following estimates hold for $(\bar{v}_q, \mathring{\bar{R}}_q)$ as well:

$$\|\bar{v}_q - v_\ell\|_0 \lesssim \delta_{q+1}^{1/2} \ell^{\theta}$$

$$\|\bar{v}_q\|_{1+N} \lesssim \delta_q^{1/2} \lambda_q \ell^{-N}$$

$$\|\mathring{\bar{R}}_q\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\theta}$$

$$\left| \int_{\mathbb{T}^3} |\bar{v}_q|^2 - |v_\ell|^2 \, dx \right| \lesssim \delta_{q+1} \ell^{\theta}$$

E.g.: $\quad \bar{v}_q - v_\ell = \sum_m \chi_m (v_m - v_\ell)$

$\|v_m - v_\ell\|_\theta \leq ?$

One has that

$$\partial_t (v_\ell - v_m) + v_\ell \cdot \nabla (v_\ell - v_m) = (v_m - v_\ell) \cdot \nabla v_m - \nabla (p_\ell - p_m) + \operatorname{div} \mathring{R}_\ell$$

In particular, using

$$\Delta(p_\ell - p_m) = \operatorname{div}(\nabla v_\ell (v_\ell - v_m)) + \operatorname{div}(\nabla v_m (v_\ell - v_m)) + \operatorname{div}\operatorname{div} \mathring{R}_\ell$$

and the estimates about elliptic operators,

$$\|\nabla (p_\ell - p_m)(\cdot,t)\|_\theta \leq (\|v_\ell\|_{1+\theta} + \|v_m\|_{1+\theta}) \|v_\ell - v_m\|_\theta + \|\mathring{R}_\ell\|_{1+\theta}$$

From the classical estimates, we have that

$\|v_m\|_{1+\theta} \leq \tau_q^{-1} \ell^{\theta} = \delta_q^{1/2} \lambda_q \ell^{-\theta}$

We also know that $\|v_\ell\|_{1+\theta} \leq \delta_q^{1/2} \lambda_q \ell^{-\theta}$

and $\|\mathring{R}_\ell\|_{1+\theta} \leq \delta_{q+1} \ell^{-1+\theta}$

Then, one has that

$$\|(\partial_t + v_\ell \cdot \nabla)(v_\ell - v_m)\|_\theta \lesssim \delta_{q+1} \ell^{-1+\theta} + \tau_q^{-1} \|v_\ell - v_m\|_\theta$$

Then, for $|t - t_m| \, \|v_\ell\|_1 \leq 1$ (which is the case now)

(Scrivi enunciato teorema classico)

$$\|v_\ell - v_m\|_\theta \leq 2\left(\underbrace{\|v_\ell - v_m\|_\theta (t_m)}_{=0} + \int_{t_m}^{t} \|(\partial_t + v_\ell \cdot \nabla)(v_\ell - v_m)\|_\alpha \right)$$

classical theorem on solutions of the transport eqns

$$\leq C\tau_q \delta_{q+1} \ell^{-1+\theta} + \int_{t_m}^{t} \tau_q^{-1} \|v_\ell - v_m\|_\theta$$

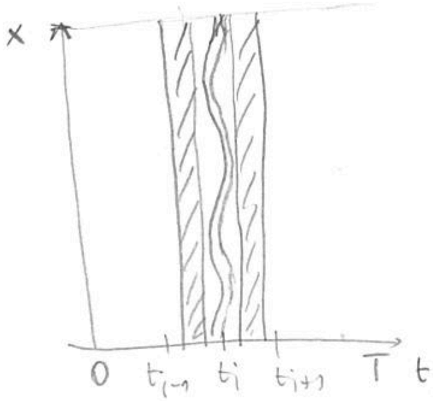
Grönwall's inequality

$$\|v_m - v_\ell\|_\alpha \lesssim \tau_q \delta_{q+1} \ell^{-1+\alpha}$$

By definition, $\delta_{q+1}^{1/2} \tau_q \ell^{-1} \leq 1$

So $\|v_m - v_\ell\|_\alpha \lesssim \delta_{q+1}^{1/2} \ell^{\alpha}$.

$$v_{q+1} = \bar{v}_q + w_{q+1}$$



The support of w_{q+1} has to include

$$\bigcup_{m\in\mathbb{N}} I_m \times \mathbb{T}^3, \qquad w_{q+1} = \sum_m w_m$$

In order to make the support of the w_m disjoint and at the same time reduce both the Reynolds stress and the energy gap we have to choose suitable cut-off functions η_m.

(explain why $\eta_m = \eta_m(t)$ not good)

Define
$$\rho_q(t) := \frac{1}{3}\left(e(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\bar{v}_q|^2 \, dx \right)$$

$$\rho_{q,m}(x,t) = \frac{\eta_m^2(x,t)}{\sum_j \int \eta_j^2(y,t)\, dy} \rho_q(t)$$

Φ_m s.t.
$$\begin{cases} (\partial_t + \bar{v}_q \cdot \nabla)\, \Phi_m = 0 \\ \Phi_m(x, t_m) = x \end{cases}$$

$$R_{q,m} := \rho_{q,m}\, Id. - \eta_m^2\, \mathring{R}_q$$

$$\tilde{R}_{q,m} = \frac{\nabla \Phi_m R_{q,m} (\nabla \Phi_m)^T}{\rho_{q,m}}$$

Thanks to $|\rho_q| \geq \delta_{q+1}$ and $\|\mathring{R}_{q}\| \leq \delta_{q+1} \ell^{\delta}$

and $\|\nabla \Phi_m - Id\| \leq \frac{1}{2}$

$$\Rightarrow \tilde{R}_{q,m} \in B_{1/2}(Id) \subset\subset S^{3\times 3}_+$$

$\sim$

the principal part of the perturbation

($w_{q+1} = w_o + w_c$ in order to be divergence free)

$$w_o = \sum_m (\rho_{q,m}(x,t))^{1/2} (\nabla\Phi_m)^{-1} W(\tilde{R}_{q,m}, \lambda_{q+1}\Phi_m) = \sum_m w_m .$$

Estimates for the Reynolds stress

Nash Error

$$\mathcal{R}(w_o \cdot \nabla \bar{v}_q) = \sum_m \mathcal{R}\Big((\rho_{q,m})(x,t)^{1/2} (\nabla\Phi_m)^{-1} W(\tilde{R}_{q,m}, \lambda_{q+1}\Phi_m) \cdot \nabla\bar{v}$$

$$\rho_{q,m}(x,t)^{1/2} (\nabla\Phi_m)^{-1} W(\tilde{R}_{q,m}, \lambda_{q+1}\Phi_m) =$$

$$= \sum_{k\neq 0} \rho_{q,m}(x,t)^{1/2} a_k(\tilde{R}_{q,m}) (\nabla\Phi_m)^{-1} A_k e^{i\lambda_{q+1} k\cdot\Phi_m}$$

$$= \sum_{k\neq 0} (\nabla\Phi_m)^{-1} b_{m,k} e^{i\lambda_{q+1} k\cdot\Phi_m}$$

$$\mathcal{R}(w_o \cdot \nabla\bar{v}_q) = \sum_m \sum_{k\neq 0} \mathcal{R}\left((\nabla\Phi_m)^{-1} b_{m,k} e^{i\lambda_{q+1} k\cdot\Phi_m} \cdot \nabla\bar{v}_q \right)$$

$$\| \mathcal{R}((\nabla\Phi_m)^{-1} b_{m,k} e^{i\lambda_{q+1} k\cdot\Phi_m} \cdot \nabla\bar{v}_q \|_\alpha$$

$$\lesssim \frac{\|\nabla\Phi_m^{-1} b_{m,k} \cdot \nabla\bar{v}_q\|_0}{\lambda_{q+1}^{1-\alpha}} + \frac{\|\nabla\Phi_m^{-1} b_{m,k}\cdot\nabla\bar{v}_q\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}}$$

$$+ \frac{\|\nabla\Phi_m^{-1} b_{m,k}\cdot\nabla\bar{v}_q\|_0 \, \|\Phi_m\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}}$$

$$\lesssim \frac{\lambda_q \delta_{q+1}^{1/2} \delta_q^{1/2}}{\lambda_{q+1}^{1-\alpha} |k|^6} + \frac{\lambda_q \delta_{q+1}^{1/2} \delta_q^{1/2}}{\lambda_q^{N-\alpha} \ell^{N+\alpha} |k|^6}$$

$$\lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}^{1-\alpha} |k|^6}$$

provided $\quad \dfrac{1}{\lambda_q^{N-\alpha} \ell^{N+\alpha}} \le \dfrac{1}{\lambda_{q+1}^{1-\alpha}}$

First choose $\alpha > 0$ small enough and then N large $\Rightarrow$ satisfied

[equivalent to:

$$\lambda_{q+1}^{(N-1)-(N-\alpha)\beta} \geq \lambda_q^{(1-\beta+3\alpha/2)(N+\alpha)}$$

$$b^{q+1}\big((N-1)-(N-\alpha)\beta\big) > b^q (N+\alpha)\left(1-\beta+\frac{3\alpha}{2}\right)$$

For α small, sufficient to check $\exists N$ s.t.

$$b\big((N-1)-N\beta\big) > N(1-\beta)$$

$$\Updownarrow$$

$$(b-1)(N-1)(1-\beta) > (1-\beta)+b\beta$$

$b>1 \quad \beta<\frac{1}{3} \quad \Rightarrow$ true if N large enough.