Poisson Geometry and Normal Forms: A Guided Tour through Examples

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From Poisson Geometry to Quantum Fields on Noncommutative Spaces, Würzburg Autumn School

Lectures 1 and 2

Outline

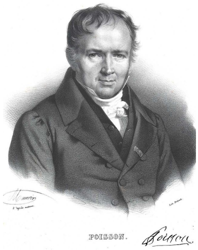
Poisson, 1809

MÉMOIRE

Sur la Variation des Constantes arbitraires dans les questions de Mécanique,

Lu à l'Institut le 16 Octobre 1809;

Par M. Poisson.



constante a ni la constante b; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais, afin de rappeler l'origine de cette quantité, qui représente une certaine combinaison des différences partielles des valeurs de a et b, nous ferons usage de cette notation (b, a), pour la désigner; de manière que nous aurons généralement

$$\frac{db}{ds}\cdot\frac{da}{d\varphi}-\frac{da}{ds}\cdot\frac{db}{d\varphi}+\frac{db}{du}\cdot\frac{da}{d\psi}-\frac{da}{du}\cdot\frac{db}{d\psi}+\frac{db}{dv}\cdot\frac{da}{d\theta}-\frac{da}{dv}\cdot\frac{db}{d\theta}=(b, a).$$

Figure: Poisson bracket

Jacobi, Lie and Lichnerowicz

Figure: Jacobi, Lie and Lichnerowicz

Example 1: Lie algebras of matrix groups

The operation on matrices $[A,B]=AB-BA$ is antisymmetric and satisfies $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0,$ (Jacobi).

Example: $SO(3,\mathbb{R})=\{A\in GL(3,\mathbb{R}), \quad A^TA=Id. \quad \det(A)=1\}$ and $\mathfrak{so}(3,R):=T_{Id}(SO(3,\mathbb{R}))=\{A\in M(3,\mathbb{R}), \quad A^T+A=0. \quad Tr(A)=0\}$.

The brackets are determined on a basis

$$e_1:=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}, \quad e_2:=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}, \quad e_3:=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$$

by $[e_1,e_2]=-e_3, \quad [e_1,e_3]=e_2, \quad [e_2,e_3]=-e_1.$
Define the (Poisson) bracket using the dual basis x_1,x_2,x_3 in $\mathfrak{so}(3,R)^*$

$$\{x_1,x_2\}=-x_3, \quad \{x_1,x_3\}=x_2, \quad \{x_2,x_3\}=-x_1$$

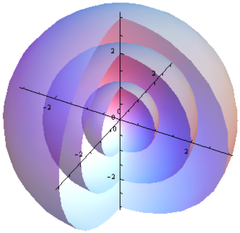
It satifies Jacobi $\{x_i,\{x_j,x_k\}\}+\{x_j,\{x_k,x_i\}\}+\{x_k,\{x_i,x_j\}\}=0$

From Lie algebras to Poisson structures (Exercise 4)

Another way to write the Poisson bracket

$$-x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$$

Using the properties of the Poisson bracket, $\{x_1^2 + x_2^2 + x_3^2, x_i\} = 0, i = 1, 2, 3$ and the function $f = x_1^2 + x_2^2 + x_3^2$ is a constant of motion.



Each sphere is endowed with an area form (symplectic structure).

Example 2: Determinants in $\mathbb{R}^3$ (Exercise 12)

- Dynamics: Given two functions $H, K \in \mathcal{C}^\infty(\mathbb{R}^3)$. Consider the system of differential equations:

$$(\dot{x}, \dot{y}, \dot{z}) = dH \wedge dK \qquad (1)$$

  H and K are constants of motion (the flow lies on $H = cte.$ and $K = cte.$)

- Geometry: Consider the brackets,

$$\{f,g\}_H := \det(df, dg, dH) \quad \{f,g\}_K := \det(df, dg, dK)$$

  They are antisymmetric and satisfy Jacobi,
  $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$
  The flow of the vector field

$$\{K,\cdot\}_H := \det(dK, \cdot, dH)$$

  and $\{-H,\cdot\}_K$ is given by the differential equation (1) and

$$\{H,K\}_H = 0, \quad \{H,K\}_K = 0$$

Example 3: Hamilton's equations

The equations of the movement of a particle can be written as Hamilton's equation using the change $p_i = \dot{q}_i$,

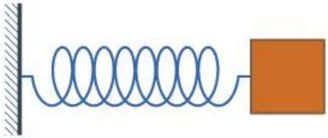
$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}$$

There is a geometrical structure behind this formula $\rightsquigarrow$ symplectic form ω (closed non-degenerate 2-form).

Non-degeneracy $\rightsquigarrow$ for every smooth function f, there exists a unique vector field X_f (Hamiltonian vector field),

$$i_{X_f}\omega = -df$$

Example 4: Coupling two simple harmonic oscillators



The phase space is $(T^*(\mathbb{R}^2), \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. H is the sum of potential and kinetic energy,

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

$H = h$ is a sphere S^3. We have rotational symmetry on this sphere $\rightsquigarrow$ the angular momentum is a constant of motion, $L = x_1 y_2 - x_2 y_1$, $X_L = (-x_2, x_1, -y_2, y_1)$ and

$$X_L(H) = \{L, H\} = 0.$$

Example 5: Cauchy-Riemann equations and Hamilton's equations

- Take a holomorphic function on $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ decompose it as $F = G + iH$ with $G, H : \mathbb{R}^4 \longrightarrow \mathbb{R}$.

  Cauchy-Riemann equations for F in coordinates $z_j = x_j + iy_j$, $j = 1, 2$

$$\frac{\partial G}{\partial x_i} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial G}{\partial y_i} = -\frac{\partial H}{\partial x_i}$$

- Reinterpret these equations as the equality

$$\{G, \cdot\}_0 = \{H, \cdot\}_1$$

  with $\{\cdot, \cdot\}_j$ the Poisson brackets associated to the real and imaginary part of the symplectic form $\omega = dz_1 \wedge dz_2$ $(\omega = \omega_0 + i\omega_1)$.
- Check $\{G, H\}_0 = 0$ and $\{H, G\}_1 = 0$ (integrable system).

Plan for today

- Definition and examples.
- Weinstein's splitting theorem and symplectic foliation.

Figure: Alan Weinstein and Reeb foliation

- Normal form theorems.

Figure: Marius Crainic, Rui Loja Fernandes and Ionut Marcut